HW 6 Due: 4 March 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. Let $G = \mathbb{Z}_{20}$ and $H = \langle 15 \rangle$. Write down all elements of all (right) cosets of H in G.
2. Let $G = D_4$ and $H = \langle R \rangle$. Write down all elements of all (right) cosets of H in G.
3. Define an equivalence relation on D_5 by setting, for all $g, h \in D_5$:

$$g \sim h \qquad \text{if and only if} \qquad \langle R \rangle\, g = \langle R \rangle\, h.$$

   For $g \in D_5$, write $[g]$ for the equivalence class of g using this relation. Write down all elements in $[F]$.

4. Define the function homomorphism

$$\begin{aligned} \phi\colon \mathbb{Z}_{10} &\to \mathbb{Z}_{10} \\ n &\mapsto n + n. \end{aligned}$$

   Let $\sim$ be the equivalence relation on $\mathbb{Z}_{10}$ obtained as in (V). From class, we know that this equivalence relation gives a partition of $\mathbb{Z}_{10}$. Write down this partition.

Proofs

(I) Let G be a group and define a relation $\sim$ on G by the condition: for all $g_1, g_2 \in G$,

$$g_1 \sim g_2 \qquad \text{if and only if} \qquad \text{there exists } h \in G \text{ such that } g_1 = h g_2 h^{-1}.$$

Prove that $\sim$ is an equivalence relation.

(II) Let G be a group, let H be a subgroup of G, and let $a, b \in G$. Prove

$$Ha = Hb \qquad \text{if and only if} \qquad ab^{-1} \in H.$$

(III) Suppose n is a positive odd integer, that G is a group of order $2n$, and that $a, b \in G$ have orders $n, 2$, respectively. Prove that

$$G = \left\{ a^i b^j \mid i \in \{0, 1\} \text{ and } j \in \{0, \dots, n-1\} \right\}.$$

(IV) Let $\mathcal{F}$ be the set of all functions with domain and codomain $\mathbb{R}$. Define $\sim$ on $\mathcal{F}$ by setting for all $f, g \in \mathcal{F}$:

$$f \sim g \qquad \text{if and only if} \qquad f(0) = g(0).$$

Prove that $\sim$ is an equivalence relation on $\mathcal{F}$.

(V) Suppose S, T are sets and $f\colon S \to T$ is a function. Define $\sim$ on S by the condition: for all $s_1, s_2 \in S$,

$$s_1 \sim s_2 \qquad \text{if and only if} \qquad f(s_1) = f(s_2).$$

Prove that $\sim$ is an equivalence relation on S.