

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

## 1 Computations

1. Consider the function  $g \in S_4$  given by

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 1 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4. \end{aligned}$$

Write down all functions  $f \in S_4$  with the property that

- (a)  $f \circ f \circ f = \text{id}_{\{1,2,3,4\}}$ ,
- (b)  $f \circ g = g \circ f$ ,
- (c)  $f \circ g = \text{id}_{\{1,2,3,4\}}$ .
- (d)  $f(1) \in \{1, 2\}$  and  $f(2) \in \{1, 2\}$ .

*Proof.* I will use cycle notation, from Chapter 8 of [Pin10]. Let's write  $\text{id}$  for  $\text{id}_{\{1,2,3,4\}}$ .

- (a)  $\text{id}, (123), (132), (134), (143), (234), (243)$ .
- (b)  $\text{id}, (12), (12)(34), (34)$ .
- (c)  $(12)$ .
- (d)  $\text{id}, (12), (12)(34), (34)$ .

□

2. How many elements  $(f, g)$  in  $S_3 \times S_3$  are there that satisfy  $(f, g)(f, g)(f, g) = (\text{id}_{\{1,2,3\}}, \text{id}_{\{1,2,3\}})$ ?

*Solution.* There are 9 such elements:

$$\begin{array}{lll} (\text{id}_{\{1,2,3\}}, \text{id}_{\{1,2,3\}}), & (\text{id}_{\{1,2,3\}}, (123)), & (\text{id}_{\{1,2,3\}}, (132)), \\ ((123), \text{id}_{\{1,2,3\}}), & ((123), (123)), & ((123), (132)), \\ ((132), \text{id}_{\{1,2,3\}}), & ((132), (123)), & ((132), (132)). \end{array}$$

□

## 2 Proofs

- (I) Define the set

$$G = \{f \in S_5 \mid f(2) = 3\}.$$

Is  $G$  a subgroup of  $S_5$ ? Prove your answer is correct.

*Proof.* Note that  $(23) \in G$  but  $(23)(23) = \text{id}_{\{1,2,3,4,5\}} \notin G$ , so  $G$  is not a subgroup of  $S_5$ . □

- (II) Suppose that  $n$  is a positive, and suppose that  $\alpha$  is a nontrivial cycle in  $S_n$ . That is, there is a  $j \in \{2, \dots, n\}$  and distinct integers  $a_1, \dots, a_j$  such that

$$\alpha = (a_1 \ a_2 \ \dots \ a_j).$$

Prove: if  $\pi$  in  $S_n$ , then  $\pi\alpha\pi^{-1}$  is the cycle

$$(\pi(a_1) \ \pi(a_2) \ \dots \ \pi(a_j)).$$

*Proof.* Let's write  $f$  for  $(\pi(a_1) \pi(a_2) \cdots \pi(a_j))$ . To show that  $\pi\alpha\pi^{-1} = f$ , we must prove: for all  $b \in \{1, 2, \dots, n\}$ , it is true that  $\pi\alpha\pi^{-1}(b) = f(b)$ . To this end, choose an arbitrary  $b \in \{1, 2, \dots, n\}$ . Since  $\pi$  is a bijection, we know there is exactly one  $c \in \{1, \dots, n\}$  such that  $b = \pi(c)$ . There are three possibilities

- if there is some  $i \in \{1, \dots, j-1\}$  such that  $c = a_i$ , then

$$\pi\alpha\pi^{-1}(b) = \pi\alpha\pi^{-1}(\pi(c)) = \pi\alpha(c) = \pi\alpha(a_i) = \pi(a_{i+1}) = f(\pi(a_i)) = f(\pi(c)) = f(b),$$

- if  $c = a_j$ , then

$$\pi\alpha\pi^{-1}(b) = \pi\alpha\pi^{-1}(\pi(c)) = \pi\alpha(c) = \pi\alpha(a_j) = \pi(a_1) = f(\pi(a_j)) = f(\pi(c)) = f(b),$$

- if  $c \notin \{a_1, \dots, a_j\}$ , then

$$\pi\alpha\pi^{-1}(b) = \pi\alpha\pi^{-1}(\pi(c)) = \pi\alpha(c) = \pi(c) = f(\pi(c)) = f(b).$$

Thus, we see that  $\pi\alpha\pi^{-1} = f$ . □

(III) Suppose that  $G, H$  are groups and  $\phi: G \rightarrow H$  is an isomorphism.

- Prove: if every element of  $G$  is its own inverse, then every element of  $H$  is its own inverse.
- Prove: if  $G$  is abelian, then  $H$  is abelian.

*Proof.* (a) Let's write  $e_G, e_H$  for the identities of  $G, H$ , respectively. Choose any  $h \in H$ . Since  $\phi$  is surjective, there is some  $g \in G$  with  $\phi(g) = h$ . Then

$$\begin{aligned} h^2 &= \phi(g)^2 && \text{(from above)} \\ &= \phi(g^2) && \text{(since } \phi \text{ respects the operations of the groups)} \\ &= \phi(e_G) && \text{(since every element of } G \text{ is its own inverse)} \\ &= e_H && \text{(fact from class).} \end{aligned}$$

- Choose any  $h_1, h_2 \in H$ . Since  $\phi$  is surjective, we know there are  $g_1, g_2 \in G$  such that  $\phi(g_1) = h_1$  and  $\phi(g_2) = h_2$ . Then

$$\begin{aligned} h_1 h_2 &= \phi(g_1) \phi(g_2) && \text{(from above)} \\ &= \phi(g_1 g_2) && \text{(since } \phi \text{ respects the operations of the groups)} \\ &= \phi(g_2 g_1) && \text{(since } G \text{ is abelian)} \\ &= \phi(g_2) \phi(g_1) && \text{(since } G \text{ is abelian)} \\ &= h_2 h_1 && \text{(from above).} \end{aligned}$$

□

(IV) Let  $m \in \mathbb{Z}_{>0}$  and let  $J$  be any subgroup of  $\mathbb{Z}_m$  of size at least 2. Prove that if  $j$  is the smallest positive integer in  $J$ , then  $\langle j \rangle = J$ . (In particular: all subgroups of  $\mathbb{Z}_m$  are cyclic.) (Recall that [Cla, Fact 3.10] tells us that  $\langle j \rangle \subseteq J$  automatically, so you only need to prove one inclusion.) (Hint: this is a long division problem.)

*Proof.* Suppose that  $k \in J$ , and perform long division to obtain  $q, r \in \mathbb{Z}_{\geq 0}$  such that

- $k = qj + r$  and
- $r < j$ .

Since  $J$  is a subgroup, we know  $-j \in J$ ; hence  $r = k + q(-j) \in J$ . But  $j$  was the smallest positive integer in  $J$ , so the fact that  $r < j$  tells us that  $r = 0$ . That is,  $k = qj \in J$ , as desired. □

## References

- [Cla] Our Group Theory Class, *Our Definition Sheet*.
- [Pin10] Charles C. Pinter, *A book of abstract algebra*, Dover Publications, Inc., Mineola, NY, 2010, Reprint of the second (1990) edition [of MR0644983]. MR. 2850284