

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1 Computations

1. Consider the function $g \in S_4$ given by

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 1 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4. \end{aligned}$$

Write down all functions $f \in S_4$ with the property that

- (a) $f \circ f \circ f = \text{id}_{\{1,2,3,4\}}$,
 - (b) $f \circ g = g \circ f$,
 - (c) $f \circ g = \text{id}_{\{1,2,3,4\}}$.
 - (d) $f(1) \in \{1, 2\}$ and $f(2) \in \{1, 2\}$.
2. How many elements (f, g) in $S_3 \times S_3$ are there that satisfy $(f, g)(f, g)(f, g) = (\text{id}_{\{1,2,3\}}, \text{id}_{\{1,2,3\}})$?

2 Proofs

- (I) Define the set

$$G = \{f \in S_5 \mid f(2) = 3\}.$$

Is G a subgroup of S_5 ? Prove your answer is correct.

- (II) Suppose that n is a positive, and suppose that α is a nontrivial cycle in S_n . That is, there is a $j \in \{2, \dots, n\}$ and distinct integers a_1, \dots, a_j such that

$$\alpha = (a_1 \ a_2 \ \cdots \ a_j).$$

Prove: if π in S_n , then $\pi\alpha\pi^{-1}$ is the cycle

$$(\pi(a_1) \ \pi(a_2) \ \cdots \ \pi(a_j)).$$

- (III) Suppose that G, H are groups and $\phi: G \rightarrow H$ is an isomorphism.

- (a) Prove: if every element of G is its own inverse, then every element of H is its own inverse.
 - (b) Prove: if G is abelian, then H is abelian.
- (IV) Let $m \in \mathbb{Z}_{>0}$ and let J be any subgroup of \mathbb{Z}_m of size at least 2. Prove that if j is the smallest positive integer in J , then $\langle j \rangle = J$. (In particular: all subgroups of \mathbb{Z}_m are cyclic.) (Recall that [Cla, Fact 3.10] tells us that $\langle j \rangle \subseteq J$ automatically, so you only need to prove one inclusion.) (Hint: this is a long division problem.)

References

[Cla] Our Group Theory Class, [Our Definition Sheet](#).