HW 2 Due: 22 January 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1 Computations

1. Write down every element of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and write down its inverse. (For one example, note that the element $(0,0)$ has inverse $-(0,0) = (0,0)$.)

2. Write down every multiple of $(1,1)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_3$.

3. Write down three elements (a,b) of $\mathbb{Z}_6 \times \mathbb{Z}_3$ with the property

$$|\{n(a,b) \mid n \in \mathbb{Z}\}| = 3.$$

4. For now and for the rest of the class, for any positive integer n, we will write

$$\mathbb{B}^n = \overbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}^{n \text{ times}}.$$

Furthermore, in this situation only, we will allow ourselves to omit commas and parentheses when writing elements of these sets; as an example, we will write $1100 \in \mathbb{B}^4$.

(a) How many elements of $\mathbb{B}^3$ have exactly two 1s?

(b) How many elements of $\mathbb{B}^4$ have exactly two 1s?

(c) How many elements of $\mathbb{B}^5$ have exactly two 1s?

(d) Let $n \in \mathbb{Z}_{\geq 2}$. Write down a formula for the number of elements of $\mathbb{B}^n$ with exactly two 1s. (No proof required.)

2 Proofs

(I) Let G be a group with identity elements e_1, e_2. Prove that $e_1 = e_2$.

(II) Let G, H be groups. Prove that if G, H are both abelian, then $G \times H$ is abelian.

(III) Let G be a group, and let $g, h \in G$. Assume that

$$\text{for all } x \in G, \text{ we have } xg = gx.$$

Prove that

$$\text{for all } x \in G, \text{ we have } x\left(hgh^{-1}\right) = \left(hgh^{-1}\right)x.$$

(IV) One might remark that for any positive integer n, every element of $\mathbb{B}^n$ is its own inverse. Prove: if G is a group with the property that every element is its own inverse, then G is abelian.

(V) Let G, H be groups with identities, e_G, e_H, respectively. Prove that $\{(e_G, h) \mid h \in H\}$ is a subgroup of $G \times H$. (A similar proof shows that $\{(g, e_H) \mid g \in G\}$ is a subgroup of $G \times H$, but you don't need to write this up.)