HW 1

DUE ON 14 JANUARY, 2026

1. Computations

(1) Let $A = \{a, b, c\}$.

  (a) Write down all functions from A to itself, until you have convinced yourself that you *could* write down all such functions. How many are there? (So the answer to this question is a number.) Here is an example of a function from A to itself:

$$a \mapsto b$$
$$b \mapsto c$$
$$c \mapsto a.$$

  (b) Write down all functions f from A to itself with the property that $f \circ f$ is the identity function on A.

  (c) Write down all functions from A to $\{1\}$.

(2) Let $B = \{0, 1, 2, 3\}$. Let C be the following set:

$$C = \{(a, b) \in B \times B \mid \text{both } a \text{ and } b \text{ are odd}\}.$$

Write down all elements of C.

2. Proofs

Here are two example problems, with solutions.

Example 2.1. For any integers a, b, define $a \star b = |ab|$. Prove $\star$ is an operation on $\mathbb{Z}$.

Proof. Choose $a, b \in \mathbb{Z}$. Note that if a and b have the same sign, then $a \star b = ab$, which is an integer, and if a and b have different signs, then $a \star b = -ab$, which is also an integer. Thus, we see that $\star$ satisfies the definition of operation. □

Here are some comments about the example.

- Literally everything in the solution is part of a complete sentence.
- Before I manipulate a, b at you, I tell you specifically what they are. (This is required when writing a proof, or I will think to myself "who are these people a and b? I don't remember being introduced to them".)
- I don't restate the definition of operation, because that is part of the set of common knowledge for this class (ie, you are writing these proofs *for me*, and I know all the definitions.)

The "easier" version of this type of question is when you must show a statement is false—you need only provide a counterexample. Often you can even skip introductions, because you will not need variables!

Example 2.2. For any $a, b \in \mathbb{Z}_{\neq 0}$, define $a \bullet b = a/b$. Prove that $\bullet$ is not an operation on $\mathbb{Z}_{\neq 0}$.

Proof. Note that 1,2 are nonzero integers but $1 \bullet 2 = 1/2$, which is not an integer. Therefore, we see that $\bullet$ is not an operation on $\mathbb{Z}_{\neq 0}$. □

Here are some exercises. Fill in the blanks *with complete sentences.*

(I) For any integers a, b, define $a \star b = a - b$. Prove $\star$ is an operation on $\mathbb{Z}$.

Proof. Choose $a, b \in \mathbb{Z}$. Note that $a \star b = a - b$, which is also an integer. __________________. □

(II) For any positive integers a, b, define $a \bullet b = a - b$. Prove $\bullet$ is not an operation on $\mathbb{Z}_{>0}$.

Proof. __________________. Therefore, we see that $\bullet$ is not an operation on $\mathbb{Z}_{>0}$. □

(III) For any $a, b \in \mathbb{Q}_{\neq 0}$, define $a \odot b = a/b$, Then $\odot$ is an operation on $\mathbb{Q}_{\neq 0}$.

Proof. Choose any $a, b \in \mathbb{Q}_{\neq 0}$. __________________. We see that

$$a \odot b = \frac{c}{d} \odot \frac{e}{f} = \frac{\frac{c}{d}}{\frac{e}{f}} = \frac{cf}{ed}.$$

Since c, d, e, f are nonzero *as we noted above*, we see that ce, df are nonzero, so that $\frac{cf}{ed} \in \mathbb{Q}_{\neq 0}$. Therefore, we conclude that $\odot$ is an operation on $\mathbb{Q}_{\neq 0}$. □

(IV) Consider for a moment the operation of multiplication on the integers. Prove that this operation admits an identity.

Proof. __________________. Thus, we see that 1 is the identity element for multiplication on the integers. □

Finally, here is an opportunity to write complete proofs:

(i) In Exercise (2), we defined the set C. For any $(a, b), (c, d) \in C$, define $(a, b) \oplus (c, d) = (a + c, b + d)$. Prove that $\oplus$ is not an operation on C.

(ii) Suppose that S, T are sets. Prove that $S \cap T = S$ if and only if $S \subseteq T$.

(iii) Suppose that m is an integer. Prove:

$$\text{if } 20 \mid m, \text{ then } 4 \mid m.$$

3. Solutions

(1) (a) There are 27 functions from A to itself:

$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$	$a\mapsto a$
$b\mapsto a$	$b\mapsto a$	$b\mapsto a$	$b\mapsto b$	$b\mapsto b$	$b\mapsto b$	$b\mapsto c$	$b\mapsto c$	$b\mapsto c$
$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$
$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$	$a\mapsto b$
$b\mapsto a$	$b\mapsto a$	$b\mapsto a$	$b\mapsto b$	$b\mapsto b$	$b\mapsto b$	$b\mapsto c$	$b\mapsto c$	$b\mapsto c$
$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$
$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$	$a\mapsto c$
$b\mapsto a$	$b\mapsto a$	$b\mapsto a$	$b\mapsto b$	$b\mapsto b$	$b\mapsto b$	$b\mapsto c$	$b\mapsto c$	$b\mapsto c$
$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$	$c\mapsto a$	$c\mapsto b$	$c\mapsto c$

(b)

$a\mapsto a$	$a\mapsto a$	$a\mapsto c$	$a\mapsto b$
$b\mapsto b$	$b\mapsto c$	$b\mapsto b$	$b\mapsto a$
$c\mapsto c$	$c\mapsto b$	$c\mapsto a$	$c\mapsto c$

(c)

$$a\mapsto 1$$
$$b\mapsto 1$$
$$c\mapsto 1$$

(2) $\{(1,1),(1,3),(3,1),(3,3)\}$

(I) "Thus, we see that $\star$ is an operation on $\mathbb{Z}$."

(II) "Note that $1 \bullet 3 = 1 - 3 = -2$ is not a positive integer."

(III) "Since $a, b \in \mathbb{Q}_{\neq 0}$, there exist $c, d, e, f \in \mathbb{Z}$, all of which are nonzero, such that $a = \frac{c}{d}$ and $b = \frac{e}{f}$."

(IV) "Recall that for any $n \in \mathbb{Z}$, we know $1 \cdot n = n \cdot 1 = n$."

(i) By Exercise (2), we know that $(1,1) \in C$, but $(1,1) \oplus (1,1) = (2,2) \notin C$. Thus, we see that $\oplus$ is not an operation on C.

(ii)
- Suppose that $S \subseteq T$. To show that $S \cap T = S$ we must show $(S \cap T) \subseteq S$ and $S \subseteq (S \cap T)$.
  - Choose any $x \in (S \cap T)$. By definition of intersection, we know $x \in S$ and $x \in T$. In particular, $x \in S$. Thus, $(S \cap T) \subseteq S$.
  - Now choose any $x \in S$. Since $S \subseteq T$, we see $x \in T$. Since $x \in S$ and $x \in T$, we know by the definition of intersection that $x \in (S \cap T)$. Thus, $S \subseteq (S \cap T)$.

  We have shown that $S \cap T = S$.
- Conversely, suppose that $S \cap T = S$. To show $S \subseteq T$, we begin by choosing any $x \in S$. Since $S \cap T = S$, we know $x \in S \cap T$. By definition of intersection, this means $x \in S$ and $x \in T$. We mention in particular: $x \in T$. Thus, we see $S \subseteq T$.

(iii) By definition of divide, there is some $k \in \mathbb{Z}$ such that $m = 20k$. But then

$$m = 20k = 4(5k),$$

so $4 \mid m$ by the definition of divide.