

## HW 1

DUE ON 14 JANUARY, 2026

### 1. COMPUTATIONS

- (1) Let  $A = \{a, b, c\}$ .
- (a) Write down all functions from  $A$  to itself, until you have convinced yourself that you *could* write down all such functions. How many are there? (So the answer to this question is a number.) Here is an example of a function from  $A$  to itself:

$$a \mapsto b$$

$$b \mapsto c$$

$$c \mapsto a.$$

- (b) Write down all functions  $f$  from  $A$  to itself with the property that  $f \circ f$  is the identity function on  $A$ .
- (c) Write down all functions from  $A$  to  $\{1\}$ .
- (2) Let  $B = \{0, 1, 2, 3\}$ . Let  $C$  be the following set:

$$C = \{(a, b) \in B \times B \mid \text{both } a \text{ and } b \text{ are odd}\}.$$

Write down all elements of  $C$ .

## 2. PROOFS

Here are two example problems, with solutions.

*Example 2.1.* For any integers  $a, b$ , define  $a \star b = |ab|$ . Prove  $\star$  is an operation on  $\mathbb{Z}$ .

*Proof.* Choose  $a, b \in \mathbb{Z}$ . Note that if  $a$  and  $b$  have the same sign, then  $a \star b = ab$ , which is an integer, and if  $a$  and  $b$  have different signs, then  $a \star b = -ab$ , which is also an integer. Thus, we see that  $\star$  satisfies the definition of operation.  $\square$

Here are some comments about the example.

- Literally everything in the solution is part of a complete sentence.
- Before I manipulate  $a, b$  at you, I tell you specifically what they are. (This is required when writing a proof, or I will think to myself “who are these people  $a$  and  $b$ ? I don’t remember being introduced to them”.)
- I don’t restate the definition of operation, because that is part of the set of common knowledge for this class (ie, you are writing these proofs *for me*, and I know all the definitions.)

The “easier” version of this type of question is when you must show a statement is false—you need only provide a counterexample. Often you can even skip introductions, because you will not need variables!

*Example 2.2.* For any nonzero integers  $a, b$ , define  $a \bullet b = a/b$ . Prove that  $\bullet$  is not an operation on  $\mathbb{Z}$ .

*Proof.* Note that 1, 2 are integers but  $1 \bullet 2 = 1/2$ , which is not an integer. Therefore, we see that  $\bullet$  is not an operation on  $\mathbb{Z}$ .  $\square$

Here are some exercises. Fill in the blanks *with complete sentences*.

- (I) For any integers  $a, b$ , define  $a \star b = a - b$ . Prove  $\star$  is an operation on  $\mathbb{Z}$ .

*Proof.* Choose  $a, b \in \mathbb{Z}$ . Note that  $a \star b = a - b$ , which is also an integer.  $\square$

- (II) For any positive integers  $a, b$ , define  $a \bullet b = a - b$ . Prove  $\bullet$  is not an operation on  $\mathbb{Z}_{>0}$ .

*Proof.*  $\square$ . Therefore, we see that  $\bullet$  is not an operation on  $\mathbb{Z}_{>0}$ .  $\square$

- (III) For any  $a, b \in \mathbb{Q}_{\neq 0}$ , define  $a \odot b = a/b$ . Then  $\odot$  is an operation on  $\mathbb{Q}_{\neq 0}$ .

*Proof.* Choose any  $a, b \in \mathbb{Q}_{\neq 0}$ .  $\square$ . We see that

$$a \odot b = \frac{c}{d} \odot \frac{e}{f} = \frac{\frac{c}{d}}{\frac{e}{f}} = \frac{ce}{df}.$$

Since  $c, d, e, f$  are nonzero *as we noted above*, we see that  $ce, df$  are nonzero, so that  $\frac{ce}{df} \in \mathbb{Q}_{\neq 0}$ . Therefore, we conclude that  $\odot$  is an operation on  $\mathbb{Q}_{\neq 0}$ .  $\square$

- (IV) Consider for a moment the operation of multiplication on the integers. Prove that this operation admits an identity.

*Proof.*  $\square$ . Thus, we see that 1 is the identity element for multiplication on the integers.  $\square$

Finally, here are opportunities to write complete proofs:

- In [Exercise \(2\)](#), we defined the set  $C$ . For any  $(a, b), (c, d) \in C$ , define  $(a, b) \oplus (c, d) = (a + c, b + d)$ . Prove that  $\oplus$  is not an operation on  $C$ .
- Suppose that  $S, T$  are sets. Prove that  $S \cap T = S$  if and only if  $S \subseteq T$ .
- Suppose that  $m$  is an integer. Prove:

$$\text{if } 20 \mid m, \text{ then } 4 \mid m.$$