

Name: \_\_\_\_\_

- Put your name in the “\_\_\_\_\_” above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

## Computations

1. Consider the following two functions, to which we will give the names  $g$  and  $h$  for this problem:

$$\begin{array}{ll}
 g: \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\} & h: \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
 1 \mapsto 3 & 1 \mapsto 3 \\
 2 \mapsto 2 & 2 \mapsto 6 \\
 3 \mapsto 4 & 3 \mapsto 3 \\
 4 \mapsto 6 & 4 \mapsto 3 \\
 5 \mapsto 7 & 5 \mapsto 8 \\
 6 \mapsto 1 & 6 \mapsto 3 \\
 7 \mapsto 9 & 7 \mapsto 5 \\
 8 \mapsto 1 & 8 \mapsto 1 \\
 9 \mapsto 5 & 9 \mapsto 9.
 \end{array}$$

- (a) Compute  $g \circ h(1)$ ,  
 (b) compute  $h \circ g(1)$ ,  
 (c) write the smallest positive integer  $n$  such that

$$\overbrace{g \circ \cdots \circ g}^{n \text{ times}}(1) = 1,$$

and

- (d) write down an element  $m$  in  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  with the property that  $h \circ h(m) = h(m)$ .

*Solution.* (a) 4,

(b) 3,

(c) 4,

(d) any of 1, 3, 4, 6, or 9.

□

2. (a) Write down three subgroups of  $D_6$  that have size 2.  
(b) Write down all elements of  $\mathbb{Z}_4 \times \mathbb{Z}_4$  that are their own inverse.  
(c) Write down three subgroups of  $\mathbb{Z}_6 \times \mathbb{Z}_6$  of size 3.  
(d) Write down any group that has at least one subgroup of size 2 and at least one subgroup of size 7.

*Solution.* (a) There are, for example:  $\langle R^3 \rangle, \langle F \rangle, \langle FR \rangle, \langle FR^2 \rangle, \langle FR^3 \rangle$ .

(b) These elements are  $(0, 0), (0, 2), (2, 0), (2, 2)$ .

(c) For example:  $\langle (2, 0) \rangle, \langle (2, 2) \rangle, \langle (0, 2) \rangle, \langle (2, 4) \rangle$ .

(d) For example:  $\mathbb{Z}_{14}$  and  $D_7$ .

□

## Proofs

(I) Let  $H = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m + 3n = 0\}$ . Prove that  $H$  is a subgroup of  $\mathbb{Z} \times \mathbb{Z}$ .

*Proof.* We will apply the subgroup test.

- Since  $2(0) + 3(0) = 0$ , we see  $(0, 0) \in H$ , so  $H \neq \emptyset$ .
- Suppose that  $(a, b), (c, d) \in H$ , so that  $2a + 3b = 0 = 2c + 3d$ . By the definition of the operation of  $\mathbb{Z} \times \mathbb{Z}$ , we know that

$$(a, b) + (c, d) = (a + c, b + d).$$

Since  $2a + 3b = 0 = 2c + 3d$ , we see that

$$2(a + c) + 3(b + d) = (2a + 3b) + (2c + 3d) = 0 + 0 = 0,$$

so  $(a, b) + (c, d) \in H$  by the definition of  $H$ .

- Suppose that  $(e, f) \in H$ , so that  $2e + 3f = 0$ . From class, we know the  $-(e, f) = (-e, -f)$ . Using the fact that  $2e + 3f = 0$ , we see that  $2(-e) + 3(-f) = -(2e + 3f) = -0 = 0$ , so  $-(e, f) \in H$ .

Thus, we conclude that  $H$  is a subgroup by the subgroup test. □

(II) Suppose that  $G$  is a group with a subgroup  $H$ . Suppose that  $g_1, g_2 \in G$  satisfy  $g_1 g_2 \in H$ . Prove:

if  $g_1 \in H$ , then  $g_2 \in H$ .

*Proof.* Since  $H$  is a subgroup and  $g_1 \in H$ , we know that  $(g_1)^{-1} \in H$ . Since  $H$  is a subgroup, it is closed under the operation of  $G$ , so  $H$  contains

$$(g_1)^{-1} (g_1 g_2) = \left( (g_1)^{-1} g_1 \right) g_2 = g_2.$$

□

(III) Define the operation  $\star$  on  $\mathbb{Z}_{\geq 3}$  by

$$m \star n = \begin{cases} m & \text{if } m \geq n \\ n & \text{if } n \geq m. \end{cases}$$

- (a) Prove that  $\star$  admits an identity element.
- (b) Either prove that  $(\mathbb{Z}_{\geq 3}, \star)$  is a group or prove that it is not a group.

*Proof.* (a) For any  $n \in \mathbb{Z}_{\geq 3}$ , we know  $3 \geq n$ ; thus,  $3 \star n = 3 = n \star 3$ , so 3 is an identity element for  $\star$ .

(b) For any  $n \in \mathbb{Z}_{\geq 3}$ , we know  $4 \star n \geq 4$ . So 4 does not have an inverse, and  $(\mathbb{Z}_{\geq 3}, \star)$  is not a group. □