Name:_____

- Put your name in the "_____" above.

- Answer all questions.

- Proofs are graded for clarity, rigor, neatness, and style.

- Good luck!

Computations

1. Consider the following two functions, to which we will give the names g and h for this problem:

$$g \colon \{1,2,3,4,5,6,7,8,9\} \to \{1,2,3,4,5,6,7,8,9\} \qquad h \colon \{1,2,3,4,5,6,7,8,9\} \to \{1,2,3,4,5,6,7,8,9\}$$

$1 \mapsto 3$	$1 \mapsto 3$
$2 \mapsto 2$	$2 \mapsto 6$
$3 \mapsto 4$	$3 \mapsto 3$
$4 \mapsto 6$	$4 \mapsto 3$
$5 \mapsto 7$	$5 \mapsto 8$
$6 \mapsto 1$	$6 \mapsto 3$
$7 \mapsto 9$	$7 \mapsto 5$
$8 \mapsto 1$	$8 \mapsto 1$
$9 \mapsto 5$	$9 \mapsto 9.$

(a) Compute $g \circ h(1)$,

(b) compute $h \circ g(1)$,

(c) write the smallest positive integer n such that

$$\overbrace{g \circ \cdots \circ g}^{n \text{ times}}(1) = 1,$$

and

(d) write down an element m in $\{1,2,3,4,5,6,7,8,9\}$ with the property that $h \circ h(m) = h(m)$.

2. (a) Write down three subgroups of D_6 that have size 2.

   (b) Write down all elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ that are their own inverse.

   (c) Write down three subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_6$ of size 3.

   (d) Write down any group that has at least one subgroup of size 2 and at least one subgroup of size 7.

Proofs

(I) Let $H = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m + 3n = 0\}$. Prove that H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(II) Suppose that G is a group with a subgroup H. Suppose that $g_1, g_2 \in G$ satisfy $g_1 g_2 \in H$. Prove:

if $g_1 \in H$, then $g_2 \in H$.

(III) Define the operation $\star$ on $\mathbb{Z}_{\geq 3}$ by

$$m \star n = \begin{cases} m & \text{if } m \geq n \\ n & \text{if } n \geq m. \end{cases}$$

(a) Prove that $\star$ admits an identity element.

(b) Either prove that $(\mathbb{Z}_{\geq 3}, \star)$ is a group or prove that it is not a group.