Math 344 Fall 2025 Final Exam

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. (a) Write down all elements of order 2 in D_6.
   (b) Write down all elements of order 3 in D_6.

Solution. (a) $F, RF, R^2F, R^3F, R^4F, R^5F, R^3$
(b) R^2, R^4

□

2. Write down

   (a) An infinite group that is not cyclic,
   (b) An infinite noncommutative group,
   (c) A group of size 81 where every element has either order 3 or order 1, and
   (d) A noncommutative group of size 88.

   Solution. Some examples are

   (a) $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$,
   (b) $\mathbb{Z} \times S_3$,
   (c) $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, and
   (d) $(\mathbb{Z}/11\mathbb{Z}) \times D_4$.

$\square$

3. Consider the group S_5, a group of size 120.

   (a) Write down any $f_1 \in S_5$ such that $|S_5/\langle f_1 \rangle| = 24$.
   (b) Write down any $f_2 \in S_5$ such that $|S_5/\langle f_2 \rangle| = 30$.
   (c) Write down any $f_3 \in S_5$ such that $|S_5/\langle f_3 \rangle| = 40$.
   (d) Write down any $f_4 \in S_5$ such that $|S_5/\langle f_4 \rangle| = 20$.

   Solution. Some options are

   (a) $f_1 = (12345)$,
   (b) $f_2 = (1234)$,
   (c) $f_3 = (123)$, and
   (d) $f_4 = (12)(345)$.

   □

Proofs

(I) Suppose that G is a group with 77 elements, that H is a group, and that $\phi\colon G \to H$ is a homomorphism. Prove that if $\ker(\phi)$ contains fewer than 7 elements, then ϕ is injective.

Proof. By Lagrange's Theorem, we know that $|\ker(\phi)|$ is a divisor of 77. Thus, we know $|\ker(\phi)|$ is either 1, 7, 11, or 77. By our hypothesis, the only option is $|\ker(\phi)| = 1$. But then, by fact from class, we know that ϕ is injective. $\square$

(II) Suppose that G is a cyclic group and H is a normal subgroup of G. Prove that G/H is cyclic.

Proof. Since G is cyclic, there is some $g_0 \in G$ with $G = \langle g_0 \rangle$. To show that G/H is cyclic, choose any $g_1 \in G$, so that Hg_1 is arbitrary in G/H. Since $G = \langle g_0 \rangle$, there is some $j \in \mathbb{Z}$ such that $g_1 = (g_0)^j$. But then we use the operation on G/H to see that $Hg_1 = H(g_0)^j = (Hg_0)^j$, so that $G/H = \langle Hg_0 \rangle$. □

(III) Suppose that G_1, G_2 are groups, and define

$$\begin{aligned} \phi\colon G_1 \times G_2 &\to G_2 \\ (g_1, g_2) &\mapsto g_2. \end{aligned}$$

In HW7, we proved

- ϕ is a homomorphism and
- $\ker(\phi)$ is isomorphic to G_1.

Prove that $(G_1 \times G_2) / \ker(\phi)$ is isomorphic to G_2.

Proof. Let's write e_1, e_2 for the identities of G_1, G_2, respectively. The desired result is immediate from the Fundamental Homomorphism Theorem, as long as we prove that ϕ is surjective. To this end, choose any $g \in G_2$ and note that $\phi(e_1, g) = g$. $\square$