HW 8 Due: 12 June 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Consider the function

$$\begin{aligned} \phi\colon (\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) &\to \mathbb{Z}/3\mathbb{Z} \\ (9\mathbb{Z}+m, 3\mathbb{Z}+n) &\mapsto 3\mathbb{Z}+n. \end{aligned}$$

   We know by HW7 that ϕ is a homomorphism. Let's write G for $(\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and H for $\ker(\phi)$.

   (a) Enumerate all elements of H

   (b) Enumerate all elements of G/H.

2. Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

Proofs

(I) Suppose G is a group and H is a normal subgroup of G. Prove: if G is abelian, then G/H is abelian.

(II) Suppose that G is a group and H is a normal subgroup of G. Let K be *any* subgroup of G, and write

$$HK = \{hk \mid h \in H \text{ and } k \in K\}.$$

   (a) Prove that HK is a subgroup of G.

   (b) Prove that H is a normal subgroup of HK.

   (c) Prove that for all $k \in K$, the coset Hk is an element of HK/H.

   (d) Define

$$\begin{aligned} \sigma\colon K &\to HK/H \\ k &\mapsto Hk. \end{aligned}$$

   Prove that σ is a surjective homomorphism.

   (e) Prove that $\ker(\sigma) = H \cap K$.

   (f) Apply the fundamental homomorphism theorem to conclude that $K/(H \cap K)$ is isomorphic to HK/H.

(III) Let G be a group with a subgroup H. Prove H is normal if and only if for all $g \in G$, we have $gH = Hg$.