HW 7 Due: 28 May 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. For any group G and element $g \in G$, define the homomorphism

$$\begin{aligned} e_{G,g} \colon \mathbb{Z} &\to G \\ n &\mapsto g^n. \end{aligned}$$

   (No need to prove that $e_{G,g}$ is a homomorphism.) Let's write $e_{G,g}(\mathbb{Z}) = \{e_{G,g}(n) \mid n \in \mathbb{Z}\}$. Enumerate elements of $e_{G,g}(\mathbb{Z})$ and find $\ker(e_{G,g})$ in the following situations:

   (a) $G = D_4, g = R$,

   (b) $G = \mathbb{Z}_{20}, g = 15$,

   (c) $G = D_4, g = F$.

   (d) $G = S_9, g = \mathrm{id}_{\{1,2,3,4,5,6,7,8,9\}}$.

2. If G is a subgroup with a subgroup H, let's write G/H for the set $\{Hg \mid g \in G\}$. Recall that G/H is a partition of G.

   (a) Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

   (b) Enumerate all elements of $D_5/\langle R \rangle$.

Proofs

(I) Suppose that G, H are groups and $\psi \colon G \to H$ is a homomorphism.

   (a) Let's write $\psi(G) = \{\psi(g) \mid g \in G\}$. Prove that $\psi(G)$ is a subgroup of H.

   (b) For J a subgroup of H, let's write $\psi^{-1}(J) = \{g \in G \mid \psi(g) \in J\}$. Prove that $\psi^{-1}(J)$ is a subgroup of G.

(II) Suppose that G is a commutative group, and let n be a positive integer. Define

$$\begin{aligned} \phi \colon G &\to G \\ g &\mapsto g^n. \end{aligned}$$

   (a) Prove that ϕ is a homomorphism of groups.

   (b) Prove that $\ker(\phi) = \{g \in G \mid \mathrm{ord}(g) \text{ is a divisor of } n\}$.

   (c) Suppose that G is finite. Suppose further that $|G|$ and n have no common factors greater than 1. Prove that ϕ is an isomorphism.

(III) Suppose that G_1, G_2 are groups, and define

$$\begin{aligned} \phi \colon G_1 \times G_2 &\to G_2 \\ (g_1, g_2) &\mapsto g_2. \end{aligned}$$

   (a) Prove that ϕ is a homomorphism of groups.

   (b) Prove that $\ker(\phi)$ is isomorphic to G_1.