HW 6 Due: 21 May 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Let $G = \mathbb{Z}_{20}$ and $H = \langle 15 \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \{0, 5, 10, 15\}$, we have

   - $H = H + 0 = \{0, 5, 10, 15\}$,
   - $H + 1 = \{1, 6, 11, 16\}$,
   - $H + 2 = \{2, 7, 12, 17\}$,
   - $H + 3 = \{3, 8, 13, 18\}$, and
   - $H + 4 = \{4, 9, 14, 19\}$.

   □

2. Let $G = D_4$ and $H = \langle R \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \left\{I, R, R^2, R^3\right\}$, we have

   - $H = HI = \left\{I, R, R^2, R^3\right\}$ and
   - $HF = \left\{F, FR^3, FR^2, FR\right\}$.

   □

3. Define an equivalence relation on D_5 by setting, for all $g, h \in D_5$:

   $$g \sim h \qquad \text{if and only if} \qquad \langle R \rangle g = \langle R \rangle h.$$

   For $g \in D_5$, write $[g]$ for the equivalence class of g using this relation. Write down all elements in $[F]$.

   Solution. Since $\langle R \rangle = \left\{I, R, R^2, R^3, ^2\right\}$, we have

   - $\langle R \rangle I = \langle R \rangle R = \langle R \rangle R^2 = \langle R \rangle R^3 = \langle R \rangle R^4 = \langle R \rangle$ and
   - $\langle R \rangle F = \langle R \rangle RF = \langle R \rangle R^2F = \langle R \rangle R^3F = \langle R \rangle R^4F = \left\{F, RF, R^2F, R^3F, R^4F\right\}$.

   Thus, we see $[F] = \left\{R, RF, R^2F, R^3F, R^4F\right\}$. □

Proofs

(I) Let G be a group, let H be a subgroup of G, and let $a, b \in G$. Prove

$$Ha = Hb \qquad \text{if and only if} \qquad ab^{-1} \in H.$$

Proof. Let e be the identity element of G.

- First, suppose $Ha = Hb$. Since $e \in H$, there exists $h \in H$ with $a = ea = hb$. Then $ab^{-1} = h \in H$.
- Conversely, suppose there is some $h_0 \in H$ with $ab^{-1} = h_0$. We prove $Ha = Hb$ in two steps:
  - To see that $Ha \subseteq Hb$, choose any $h \in H$ and note that $ha = h(h_0 b) = (hh_0) b \in Hb$.
  - For the other inclusion, choose any $h \in H$ and note that $hb = h\left((h_0)^{-1} a\right) = \left(h (h_0)^{-1}\right) a \in Ha$.

□

(II) Suppose n is a positive odd integer, that G is a group of order $2n$, and that $a, b \in G$ have orders $2, n$, respectively. Prove that

$$G = \left\{ a^i b^j \mid i \in \{0,1\} \text{ and } j \in \{0,\dots,n-1\} \right\}.$$

Proof. Since $\operatorname{ord}(b) = n$, we know that $|\langle b\rangle| = n$, so that $[G : \langle b\rangle] = \frac{2n}{n} = 2$. That is, there are *precisely two* cosets of $\langle b\rangle$ in G. By Lagrange's Theorem, the fact that n is odd and $\operatorname{ord}(a) = 2$ tells us that $a \notin \langle b\rangle$, so the two cosets of $\langle b\rangle$ that partition G are:

- $\langle b\rangle = \left\{ b^0, \dots, b^{n-1} \right\}$ and
- $a\langle b\rangle = \left\{ ab^0, \dots, ab^{n-1} \right\}$.

□

(III) Let $\mathcal{F}$ be the set of all functions with domain and codomain $\mathbb{R}$. Define $\sim$ on $\mathcal{F}$ by setting for all $f, g \in \mathcal{F}$:

$$f \sim g \qquad \text{if and only if} \qquad f(0) = g(0).$$

Prove that $\sim$ is an equivalence relation on $\mathcal{F}$.

Proof. We must prove three things.

- To prove $\sim$ is reflexive, choose any $f \in \mathcal{F}$. Since $f(0) = f(0)$, we see $f \sim f$.
- To prove $\sim$ is symmetric, choose any $f, g \in \mathcal{F}$ and assume that $f \sim g$. By definition of $\sim$, this means that $f(0) = g(0)$. Well then, this means $g(0) = f(0)$ too, so that $g \sim f$.
- Finally, to prove $\sim$ is transitive, choose any $f, g, h \in \mathcal{F}$ and assume that $f \sim g$ and $g \sim h$. By definition of $\sim$, this means that $f(0) = g(0)$ and $g(0) = h(0)$. But then, by transitivity of equality, we see that $f(0) = h(0)$, so that $f \sim h$.

□