HW 4 Solutions

1 Computations

1. Consider the function $g \in S_4$ given by

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 1 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4. \end{aligned}$$

   Write down all functions $f \in S_4$ with the property that

   (a) $f \circ f \circ f = \mathrm{id}_A$,
   (b) $f \circ g = g \circ f$,
   (c) $f \circ g = \mathrm{id}_{\{1,2,3,4\}}$.
   (d) $f(1) \in \{1,2\}$ and $f(2) \in \{1,2\}$.

   Solution. I will use the very-convenient cycle notation from Chapter 8 of [Pin10].

   (a) $\mathrm{id}_A, (123), (132), (134), (143), (234), (243)$.
   (b) $\mathrm{id}_A, (12), (12)(34), (34)$.
   (c) (12).
   (d) $\mathrm{id}_A, (12), (12)(34), (34)$.

   □

2. How many elements (f, g) in $S_3 \times S_3$ are there that satisfy $(f,g)(f,g)(f,g) = \left(\mathrm{id}_{\{1,2,3\}}, \mathrm{id}_{\{1,2,3\}}\right)$?

   Solution. There are 9 such elements:

$$\begin{array}{lll} \left(\mathrm{id}_{\{1,2,3\}}, \mathrm{id}_{\{1,2,3\}}\right), & \left(\mathrm{id}_{\{1,2,3\}}, (123)\right), & \left(\mathrm{id}_{\{1,2,3\}}, (132)\right), \\ \left((123), \mathrm{id}_{\{1,2,3\}}\right), & ((123), (123)), & ((123), (132)), \\ \left((132), \mathrm{id}_{\{1,2,3\}}\right), & ((132), (123)), & ((132), (132)). \end{array}$$

   □

2 Proofs

(I) Let A be a set and B any *finite* subset of A. Define

$$G = \{f \in S_A \mid \text{for all } b \in B,\ f(b) \in B\}.$$

Prove that G is a subgroup of S_A.

Proof.

- Since $\mathrm{id}_A \in G$, we see that $G \neq \varnothing$.
- Choose $f, g \in G$. To see that $f \circ g \in B$, choose any $b \in B$. Since $g \in G$, we know that $g(b) \in B$; and since $f \in G$, we know that $f(g(b)) \in B$. Thus, we see that $f \circ g \in G$ by the definition of G.
- Finally, suppose that $f \in G$. By definition of G, we know that $f(B) \subseteq B$, but since B is finite and f is (in particular) injective, we know $f(B) = B$.
  Thus, for any $b \in B$, there exists $b_0 \in B$ such that $f(b_0) = b$. In other words, we see that $f^{-1}(b) = b_0 \in B$. Since we chose b arbitrarily, we see $f^{-1} \in G$ by definition of G.

□

(II) Define the set

$$G = \{f \in S_5 \mid f(2) = 3\}.$$

Is G a subgroup of S_5? Prove your answer is correct.

Proof. By a lemma from class we know any subgroup of S_5 contains $\mathrm{id}_{\{1,2,3,4,5\}}$. But

$$\mathrm{id}_{\{1,2,3,4,5\}}(2) = 2 \neq 3,$$

so $\mathrm{id}_{\{1,2,3,4,5\}} \notin G$. Thus, we conclude that G is not a subgroup of S_5. □

(III) Let G be a group, and define

$$D = \{(g, g) \mid g \in G\}.$$

Assume that D is a subgroup of $G \times G$.[1] Prove that G is isomorphic to D.

Proof. Define

$$\begin{aligned} \phi\colon G &\to D \\ g &\mapsto (g, g). \end{aligned}$$

- To see that ϕ is surjective, choose any $g \in G$, so that $(g, g) \in D$ is arbitrary. But then we see $\phi(g) = (g, g)$.
- To see that ϕ is injective, choose any $g, h \in G$ and assume that $\phi(g) = \phi(h)$. But then $(g, g) = \phi(g) = \phi(h) = (h, h)$, so that $g = h$.
- To see that ϕ respects the operations of G, D, choose any $g, h \in G$ and note that $\phi(gh) = (gh, gh) = (g, g)(h, h) = \phi(g)\phi(h)$.

□

References

[Pin10] Charles C. Pinter, *A book of abstract algebra*, Dover Publications, Inc., Mineola, NY, 2010, Reprint of the second (1990) edition [of MR0644983]. MR 2850284

[1] This is always true and in principle, we could prove it. However, I'm not requiring a proof. At this time.