HW 2 Solutions Due: 16 April 2025

1 Computations

1. Write down every element of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and write down its inverse. (For one example, note that the element $(0,0)$ has inverse $-(0,0) = (0,0)$.)
2. Write down every multiple of $(1,1)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_3$.
3. Write down three elements (a,b) of $\mathbb{Z}_6 \times \mathbb{Z}_3$ with the property

$$|\{n(a,b) \mid n \in \mathbb{Z}\}| = 3.$$

Solution.

1.
   - $(0,0)$ has inverse $-(0,0) = (0,0)$,
   - $(1,0)$ has inverse $-(1,0) = (2,0)$,
   - $(2,0)$ has inverse $-(2,0) = (1,0)$,
   - $(0,1)$ has inverse $-(0,1) = (0,2)$,
   - $(1,1)$ has inverse $-(1,1) = (2,2)$,
   - $(2,1)$ has inverse $-(2,1) = (1,2)$,
   - $(0,2)$ has inverse $-(0,2) = (0,1)$,
   - $(1,2)$ has inverse $-(1,2) = (2,1)$, and
   - $(2,2)$ has inverse $-(2,2) = (1,1)$.
2. $\{n(1,1) \mid n \in \mathbb{Z}\} = \{(1,1),(2,2),(3,0),(4,1),(5,2),(0,0)\}$.
3. There are 8 such elements: $(2,0),(4,0),(0,1),(2,1),(4,1),(0,2),(2,2),(4,2)$.

□

2 Proofs

(I) Let G be a group with identity elements e_1, e_2. Prove that $e_1 = e_2$.

(II) Let G, H be groups. Prove that if G, H are both abelian, then $G \times H$ is abelian.

(III) Let G be a group, and let $g, h \in G$. Assume that

$$\text{for all } x \in G, \text{ we have } xg = gx.$$

Prove that

$$\text{for all } x \in G, \text{ we have } x\left(hgh^{-1}\right) = \left(hgh^{-1}\right)x.$$

Solutions. (I) Note that by the definition of identity element,

$$\begin{aligned} e_1 &= e_1 e_2 && (e_2 \text{ is an identity, so we scale } e_1 \text{ by } e_2 \text{ on the right}) \\ &= e_2 && (e_1 \text{ is \textit{also} an identity, so we scale } e_2 \text{ by } e_1 \text{ on the left}). \end{aligned}$$

(II) Choose any $(a, b), (c, d) \in G \times H$. Note that

$$\begin{aligned} (a, b)(c, d) &= (ac, bd) && (\text{definition of products of groups}) \\ &= (ca, db) && (\text{both } G \text{ and } H \text{ are abelian}) \\ &= (c, d)(a, b) && (\text{definition of products of groups}). \end{aligned}$$

(III) Let's write e for the identity element of G. Choose any $x \in G$ and note that

$$\begin{aligned} x\left(hgh^{-1}\right) &= ((xh)\,g)\,h^{-1} && (\text{associativity}) \\ &= (g\,(xh))\,h^{-1} && (\text{by hypothesis, applied to } (xh)) \\ &= (gx)\left(hh^{-1}\right) && (\text{associativity}) \\ &= gx && (\text{definition of inverse}) \\ &= gex && (\text{definition of identity}) \\ &= g\left(hh^{-1}\right)x && (\text{definition of inverse}) \\ &= (gh)\left(h^{-1}x\right) && (\text{associativity}) \\ &= (hg)\left(h^{-1}x\right) && (\text{by hypothesis, applied to } h) \\ &= \left(hgh^{-1}\right)x && (\text{associativity}). \end{aligned}$$

□