Math 344 — Spring 2025 — Midterm

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. For the following element f of S_9, do the following:

   (a) write f in disjoint cycle form,

   (b) write f as a product of transpositions,

   (c) state the parity of f, and

   (d) write $(126) \circ (389) \circ f$ in disjoint cycle form.

$$
\begin{gathered}
f\colon \{1,2,3,4,5,6,7,8,9\} \to \{1,2,3,4,5,6,7,8,9\} \\
1 \mapsto 7 \\
2 \mapsto 2 \\
3 \mapsto 4 \\
4 \mapsto 5 \\
5 \mapsto 3 \\
6 \mapsto 1 \\
7 \mapsto 8 \\
8 \mapsto 9 \\
9 \mapsto 6
\end{gathered}
$$

2. Suppose that G is a group with a subgroup H. For any $g \in G$, define $gH = \{gh \mid h \in H\}$. In disjoint cycle form, enumerate all elements of gH in the following situations:

   (a) $G = S_3$, $H = \langle (123) \rangle$, $g = (132)$,

   (b) $G = S_3$, $H = \langle (123) \rangle$, $g = (12)$, and

   (c) $G = S_3$, $H = \langle (123) \rangle$, $g = (23)$.

Proofs

(I) Let $H = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid \text{ both } m \text{ and } n \text{ are even}\}$.

(a) Prove that H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(b) Prove that $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to H

(II) Suppose that G, H are groups and that $\phi\colon G \to H$ is an isomorphism. Prove: if G is commutative, then H is commutative.

(III) Suppose that G is a group, say with identity e. For any $g \in G$, define

$$f_g: G \to G$$
$$x \mapsto gx.$$

We have shown that $f_g \in S_G$. (You don't need to do this again.) Prove:

$$\text{for all } g \in G, \text{ if } f_g \text{ is an isomorphism, then } g = e.$$

Extra Credit (if you have extra time)

Suppose that G is a group with two subgroups I and J. Prove that

$$I \cup J \text{ is a subgroup of } G \qquad \text{if and only if} \qquad \text{either } I \subseteq J \text{ or } J \subseteq I.$$