Math 344 Spring 2024 Final

Computations

1. (a) Write down all elements of order 2 in D_6.
   (b) Write down all elements of order 3 in D_6.

   Solution. (a) $F, RF, R^2F, R^3F, R^4F, R^5F, R^3$
   (b) R^2, R^4

   □

2. Write down
   (a) An infinite group that is not cyclic,
   (b) An infinite noncommutative group,
   (c) A group of size 81 where every element has either order 3 or order 1, and
   (d) A noncommutative group of size 88.

   Solution. Some examples are
   (a) $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$,
   (b) $\mathbb{Z} \times S_3$,
   (c) $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, and
   (d) $(\mathbb{Z}/11\mathbb{Z}) \times D_4$.

   □

3. For the following element f of S_9, do the following:
   (i) write f in disjoint cycle form,
   (ii) write f as a product of transpositions, and
   (iii) write $(136) \circ (369) \circ f$ in disjoint cycle form.

$$
\begin{aligned}
f\colon \{1,2,3,4,5,6,7,8,9\} &\to \{1,2,3,4,5,6,7,8,9\} \\
1 &\mapsto 4 \\
2 &\mapsto 7 \\
3 &\mapsto 2 \\
4 &\mapsto 8 \\
5 &\mapsto 3 \\
6 &\mapsto 1 \\
7 &\mapsto 6 \\
8 &\mapsto 5 \\
9 &\mapsto 9
\end{aligned}
$$

   Solution. (a) (14853276)
   (b) (16)(17)(12)(13)(15)(18)(14)
   (c) (1485)(27963)

   □

Proofs

(I) Suppose that G is a group with 77 elements, that H is any group, and that $\phi: G \to H$ is a homomorphism. Prove that if $\ker(\phi)$ contains fewer than 7 elements, then ϕ is injective.

Proof. By Lagrange's Theorem, we know that $|\ker(\phi)|$ is a divisor of 77. Thus, we know $|\ker(\phi)|$ is either 1, 7, 11, or 77. By our hypothesis, the only option is $|\ker(\phi)| = 1$. But then, by fact from class, we know that ϕ is injective. □

(II) Suppose that G is a cyclic group and H is a normal subgroup of G. Prove that G/H is cyclic.

Proof. Since G is cyclic, there is some $g_0 \in G$ with $G = \langle g_0 \rangle$. To show that G/H is cyclic, choose any $g_1 \in G$, so that Hg_1 is arbitrary in G/H. Since $G = \langle g \rangle$, there is some $j \in \mathbb{Z}$ such that $g_1 = (g_0)^j$. But then we use the operation on G/H to see that $Hg_1 = H(g_0)^j = (Hg_0)^j$, so that $G/H = \langle Hg_0 \rangle$. □

(III) Let $G = (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and $H = \langle (6\mathbb{Z}+3, 3\mathbb{Z}+1) \rangle$.

$$\begin{aligned} \phi: \mathbb{Z} &\to (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \\ a &\mapsto (6\mathbb{Z}+3a, 3\mathbb{Z}+a) \end{aligned}$$

(a) Prove that ϕ is a homomorphism.

(b) Prove that the range of ϕ is H.

(c) Prove that $\ker(\phi) = 6\mathbb{Z}$.

(d) Apply the Fundamental Homomorphism Theorem to deduce that H is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

Proof. (a) Choose any $a, b \in \mathbb{Z}$ and note that

$$\begin{aligned} \phi(a+b) &= (6\mathbb{Z}+3(a+b), 3\mathbb{Z}+(a+b)) && \text{(definition of } \phi) \\ &= (6\mathbb{Z}+(3a+3b), 3\mathbb{Z}+(a+b)) && \text{(addition on } \mathbb{Z} \text{ is commutative)} \\ &= ((6\mathbb{Z}+3a)+(6\mathbb{Z}+3b), (3\mathbb{Z}+a)+(3\mathbb{Z}+b)) && \text{(the operations on } \mathbb{Z}/6\mathbb{Z} \text{ and } \mathbb{Z}/3\mathbb{Z}) \\ &= (6\mathbb{Z}+3a, 3\mathbb{Z}+a)+(6\mathbb{Z}+3b, 3\mathbb{Z}+b) && \text{(the operation on } (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})) \\ &= \phi(a)+\phi(b) && \text{(definition of } \phi.) \end{aligned}$$

(b) Choose any integer j, so that $j(6\mathbb{Z}+3, 3\mathbb{Z}+1) = (6\mathbb{Z}+3j, 3\mathbb{Z}+j)$ is arbitrary in H. So then $\phi(j) = (6\mathbb{Z}+3j, 3\mathbb{Z}+j) = j(6\mathbb{Z}+3, 3\mathbb{Z}+1)$, and we see the range of ϕ contains H. Conversely, choose any $j \in \mathbb{Z}$ and note $\phi(j) = (6\mathbb{Z}+3j, 3\mathbb{Z}+j) = j(6\mathbb{Z}+3, 3\mathbb{Z}+1) \in H$, so that the range of ϕ is contained in H.

(c) Choose any integer j, so $6j$ is arbitrary in $6\mathbb{Z}$. So then $\phi(6j) = (6\mathbb{Z}+3(6j), 3\mathbb{Z}+(6j)) = (6\mathbb{Z}+0, 3\mathbb{Z}+0)$ since $18j \in 6\mathbb{Z}$ and $6j \in 3\mathbb{Z}$. Thus we see that $6\mathbb{Z} \subseteq \ker(\phi)$. Conversely, choose any $j \in \ker(\phi)$, so that $(6\mathbb{Z}+0, 3\mathbb{Z}+0) = \phi(j) = (6\mathbb{Z}+3j, 3\mathbb{Z}+j)$. Then we see that $6 \mid 3j$ and $3 \mid j$, so j is even and divisible by 3. Thus, we see $6 \mid j$, so $j \in 6\mathbb{Z}$. That is, we see $\ker(\phi) \subseteq 6\mathbb{Z}$.

(d) By the Fundamental Homomorphism Theorem and (a)–(c), we know

$$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/\ker(\phi) \simeq \operatorname{im}(\phi) = H$$

□