Math 344 Spring 2024 Final

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. (a) Write down all elements of order 2 in D_6.
   (b) Write down all elements of order 3 in D_6.

2. Write down

   (a) An infinite group that is not cyclic,
   (b) An infinite noncommutative group,
   (c) A group of size 81 where every element has either order 3 or order 1, and
   (d) A noncommutative group of size 88.

3. For the following element f of S_9, do the following:

(i) write f in disjoint cycle form,

(ii) write f as a product of transpositions, and

(iii) write $(136) \circ (369) \circ f$ in disjoint cycle form.

$$
\begin{aligned}
f \colon \{1,2,3,4,5,6,7,8,9\} &\to \{1,2,3,4,5,6,7,8,9\} \\
1 &\mapsto 4 \\
2 &\mapsto 7 \\
3 &\mapsto 2 \\
4 &\mapsto 8 \\
5 &\mapsto 3 \\
6 &\mapsto 1 \\
7 &\mapsto 6 \\
8 &\mapsto 5 \\
9 &\mapsto 9
\end{aligned}
$$

Proofs

(I) Suppose that G is a group with 77 elements, that H is any group, and that $\phi: G \to H$ is a homomorphism. Prove that if $\ker(\phi)$ contains fewer than 7 elements, then ϕ is injective.

(II) Suppose that G is a cyclic group and H is a normal subgroup of G. Prove that G/H is cyclic.

(III) Let $G = (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and $H = \langle (6\mathbb{Z} + 3, 3\mathbb{Z} + 1) \rangle$.

$$\begin{aligned} \phi \colon \mathbb{Z} &\to (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \\ a &\mapsto (6\mathbb{Z} + 3a, 3\mathbb{Z} + a) \end{aligned}$$

(a) Prove that ϕ is a homomorphism.

(b) Prove that the range of ϕ is H.

(c) Prove that $\ker(\phi) = 6\mathbb{Z}$.

(d) Apply the Fundamental Homomorphism Theorem to deduce that H is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.