HW 8 Due: 5 June 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. Consider the function

$$\begin{aligned}\phi\colon (\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) &\to \mathbb{Z}/3\mathbb{Z}\\ (9\mathbb{Z}+m, 3\mathbb{Z}+n) &\mapsto 3\mathbb{Z}+n.\end{aligned}$$

   We know by Exercise (IV)(a) that ϕ is a homomorphism of groups. Let's write G for $(\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and H for $\ker(\phi)$.

   (a) Enumerate all elements of H

   (b) Enumerate all elements of G/H.

   Solution. (a)

$$H = \{(9\mathbb{Z}, 3\mathbb{Z}), (9\mathbb{Z}+1, 3\mathbb{Z}), (9\mathbb{Z}+2, 3\mathbb{Z}), (9\mathbb{Z}+3, 3\mathbb{Z}), (9\mathbb{Z}+4, 3\mathbb{Z}), (9\mathbb{Z}+5, 3\mathbb{Z}), (9\mathbb{Z}+6, 3\mathbb{Z}), (9\mathbb{Z}+7, 3\mathbb{Z}), (9\mathbb{Z}+8, 3\mathbb{Z})\}$$

   (b)

$$G/H = \{H, H + (9\mathbb{Z}, 3\mathbb{Z}+1), H + (9\mathbb{Z}, 3\mathbb{Z}+2)\}$$

   □

2. Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

   Solution. $4\mathbb{Z}/8\mathbb{Z} = \{8\mathbb{Z}+0, 8\mathbb{Z}+4\}$. □

Proofs

(I) Suppose that G is a commutative group, and let n be a positive integer. Define

$$\begin{aligned}\phi\colon G &\to G\\ g &\mapsto g^n.\end{aligned}$$

   (a) Prove that ϕ is a homomorphism of groups.

   (b) Prove that $\ker(\phi) = \{g \in G \mid \operatorname{ord}(g) \text{ is a divisor of } n\}$.

   (c) Suppose that G is finite. Suppose further that $|G|$ and n have no common factors greater than 1. Prove that ϕ is an isomorphism.

   Proof. Let's write e for the identity of G.

   (a) Choose any $g, h \in G$, and note that

$$\begin{aligned}\phi(gh) &= (gh)^n\\ &= g^n h^h && \text{(since } G \text{ is commutative)}\\ &= \phi(g)\phi(h),\end{aligned}$$

   so we see that ϕ is a homomorphism.

(b) We must show two inclusions.

- Suppose that $g \in \ker(\phi)$, so that $g^n = \phi(g) = e$. By fact from class, we know that $\mathrm{ord}(g)$ is a divisor of n.
- Now suppose that $g \in G$ and $\mathrm{ord}(g)$ is a divisor of n: let's say there's some $m \in \mathbb{Z}_{>0}$ with $n = \mathrm{ord}(g) \cdot m$. Then $\phi(g) = g^n = \left(g^{\mathrm{ord}(g)}\right)^m = e^m = e$, so $g \in \ker(\phi)$.

(c) Since G is finite, we need only show that ϕ is injective to deduce that it is bijective; the result will then follow from part (a). Well, from class we know we can show ϕ is injective by showing $\ker(\phi) = \{e\}$; to this end, suppose that $g \in \ker(\phi)$. Then $g^n = \phi(g) = e$, so that $\mathrm{ord}(g)$ is a divisor of n. But by Lagrange's theorem, we know that $\mathrm{ord}(g)$ is *also* a divisor of $|G|$. By hypothesis, we see that $\mathrm{ord}(g) = 1$; that is, we see that $g = e$, as desired.

□

(II) Suppose G is a group and H is a normal subgroup of G. Prove: if G is abelian, then G/H is abelian.

Proof. Suppose that $a, b \in G$ and note that since G is commutative:

$$(Ha)(Hb) = H(ab) = H(ba) = (Hb)(Ha).$$ □

(III) Let G be a group with a subgroup H. Prove H is normal if and only if for all $g \in G$, we have $gH = Hg$.

Proof.
- First let's suppose that H is normal. Choose any $g \in G$.
  - Choose any $h \in H$, so that $gh \in Hg$ is arbitrary. The fact that H is a normal subgroup tells us that $gh = gh\left(g^{-1}g\right) = \left(ghg^{-1}\right)g \in Hg$.
  - Choose any $h \in H$, so that $hg \in Hg$ is arbitrary. The fact that H is a normal subgroup tells us that $hg = \left(gg^{-1}\right)hg = g\left(g^{-1}hg\right) \in gH$.
- Now suppose that for all $g \in G$, we have $gH = Hg$. To show that H is normal, choose any $g \in G$ and $h \in H$. Use our hypothesis to find an $h_0 \in H$ with $gh = h_0 g$ and note that $ghg^{-1} = (h_0 g)g^{-1} = h_0 \in H$.

□

(IV) Suppose that G_1, G_2 are groups, and define

$$\begin{aligned}\phi\colon G_1 \times G_2 &\to G_2\\ (g_1, g_2) &\mapsto g_2.\end{aligned}$$

(a) Prove that ϕ is a homomorphism of groups.

(b) Prove that $\ker(\phi)$ is isomorphic to G_1.

Proof. (a) Choose any $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$ and note that

$$\phi\left((g_1, g_2)(h_1, h_2)\right) = \phi(g_1 h_1, g_2 h_2) = g_2 h_2 = \phi(g_1, g_2)\,\phi(g_1, g_2).$$

(b) Let's write e_1, e_2 for the identities of G_1, G_2, respectively. I claim that for any $g_1 \in G_1$, the element (g_1, e_2) is in $\ker(\phi)$; indeed, note that $\phi(g_1, e_2) = e_2$ by the definition of ϕ. Thus, we may define

$$\begin{aligned}\psi\colon G_1 &\to \ker(\phi)\\ g_1 &\mapsto (g_1, e_2).\end{aligned}$$

- To see that ψ is a homomorphism, choose any $g_1, h_1 \in G_1$ and note that

$$\psi(g_1 h_1) = (g_1 h_1, e_2) = (g_1, e_2)(h_1, e_2) = \psi(g_1)\,\psi(h_1).$$

- To see that ψ is injective, choose any $g_1 \in \ker(\psi)$. Then $(e_1, e_2) = \psi(g_1) = (g_1, e_2)$, so we see $g_1 = e_1$. By fact from class, this means that ψ is injective.
- Finally, to see that ψ is surjective, choose any $(g_1, g_2) \in \ker(\phi)$. This means that $e_2 = \phi(g_1, g_2) = g_2$, so that

$$\psi(g_1) = (g_1, e_2) = (g_1, g_2).$$ □