HW 8 Due: 5 June 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Consider the function

$$\begin{aligned} \phi\colon (\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) &\to \mathbb{Z}/3\mathbb{Z} \\ (9\mathbb{Z} + m, 3\mathbb{Z} + n) &\mapsto 3\mathbb{Z} + n. \end{aligned}$$

   We know by Exercise (IV)(a) that ϕ is a homomorphism of groups. Let's write G for $(\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and H for $\ker(\phi)$.

   (a) Enumerate all elements of H

   (b) Enumerate all elements of G/H.

2. Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

Proofs

(I) Suppose that G is a commutative group, and let n be a positive integer. Define

$$\begin{aligned} \phi\colon G &\to G \\ g &\mapsto g^n. \end{aligned}$$

   (a) Prove that ϕ is a homomorphism of groups.

   (b) Prove that $\ker(\phi) = \{g \in G \mid \operatorname{ord}(g) \text{ is a divisor of } n\}$.

   (c) Suppose that G is finite. Suppose further that $|G|$ and n have no common factors greater than 1. Prove that ϕ is an isomorphism.

(II) Suppose G is a group and H is a normal subgroup of G. Prove: if G is abelian, then G/H is abelian.

(III) Let G be a group with a subgroup H. Prove H is normal if and only if for all $g \in G$, we have $gH = Hg$.

(IV) Suppose that G_1, G_2 are groups, and define

$$\begin{aligned} \phi\colon G_1 \times G_2 &\to G_2 \\ (g_1, g_2) &\mapsto g_2. \end{aligned}$$

   (a) Prove that ϕ is a homomorphism of groups.

   (b) Prove that $\ker(\phi)$ is isomorphic to G_1.