HW 7 Due: 29 May 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Let $G = \mathbb{Z}_{20}$ and $H = \langle 15 \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \{0, 5, 10, 15\}$, we have

   - $H = H + 0 = \{0, 5, 10, 15\}$,
   - $H + 1 = \{1, 6, 11, 16\}$,
   - $H + 2 = \{2, 7, 12, 17\}$,
   - $H + 3 = \{3, 8, 13, 18\}$, and
   - $H + 4 = \{4, 9, 14, 19\}$.

   $\square$

2. Let $G = D_4$ and $H = \langle R \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \left\{I, R, R^2, R^3\right\}$, we have

   - $H = HI = \left\{I, R, R^2, R^3\right\}$ and
   - $HF = \left\{F, FR^3, FR^2, FR\right\}$.

   $\square$

3. For any group G and element $g \in G$, define the homomorphism

$$\begin{aligned} e_{G,g} \colon \mathbb{Z} &\to G \\ n &\mapsto g^n. \end{aligned}$$

   (No need to prove that $e_{G,g}$ is a homomorphism.) Let's write $e_{G,g}(\mathbb{Z}) = \{e_{G,g}(n) \mid n \in \mathbb{Z}\}$. Enumerate elements of $e_{G,g}(\mathbb{Z})$ and find $\ker(e_{G,g})$ in the following situations:

   (a) $G = D_4, g = R$,

   (b) $G = \mathbb{Z}_{20}, g = 15$,

   (c) $G = D_4, g = F$.

   (d) $G = S_9, g = \mathrm{id}_{\{1,2,3,4,5,6,7,8,9\}}$.

   Solution. (a) $e_G, g(\mathbb{Z}) = \left\{I, R, R^2, R^3\right\}$, $\ker(e_G, g) = 4\mathbb{Z}$,

   (b) $e_G, g(\mathbb{Z}) = \{0, 5, 10, 15\}$, $\ker(e_G, g) = 4\mathbb{Z}$,

   (c) $e_G, g(\mathbb{Z}) = \{I, F\}$, $\ker(e_G, g) = 2\mathbb{Z}$, and

   (d) $e_G, g(\mathbb{Z}) = \left\{\mathrm{id}_{\{1,2,3,4,5,6,7,8,9\}}\right\}$, $\ker(e_G, g) = \mathbb{Z}$.

   $\square$

Proofs

(I) Let G be a group, let H be a subgroup of G, and let $a, b \in G$. Prove

$$Ha = Hb \qquad \text{if and only if} \qquad ab^{-1} \in H.$$

Proof. Let e be the identity element of G.

- First, let's suppose that $Ha = Hb$. Since $e \in H$, there is some $h \in H$ with $a = ea = hb$. Then we see $ab^{-1} = h \in H$.
- Conversely, suppose there is some $h_0 \in H$ with $ab^{-1} = h_0$. We prove $Ha = Hb$ in two steps:
  - To see that $Ha \subseteq Hb$, choose any $h \in H$ and note that $ha = h(h_0 b) = (hh_0) b \in Hb$.
  - For the other inclusion, choose any $h \in H$ and note that $hb = h\left((h_0)^{-1} a\right) = \left(h (h_0)^{-1}\right) a \in Ha$.

□

(II) Suppose n is a positive odd integer, that G is a group of order $2n$, and that $a, b \in G$ have orders $2, n$, respectively. Prove that

$$G = \left\{a^i b^j \mid i \in \{0,1\} \text{ and } j \in \{0, \dots, n-1\}\right\}.$$

Proof. Since $\operatorname{ord}(b) = n$, we know that $|\langle b\rangle| = n$, so that $[G : \langle b\rangle] = \frac{2n}{n} = 2$. That is, there are *precisely two* cosets of $\langle b\rangle$ in G. By Lagrange's Theorem, the fact that n is odd and $\operatorname{ord}(a) = 2$ tells us that $a \notin \langle b\rangle$, so the two cosets of $\langle b\rangle$ that partition G are:

- $\langle b\rangle = \left\{b^0, \dots, b^{n-1}\right\}$ and
- $a\langle b\rangle = \left\{ab^0, \dots, ab^{n-1}\right\}$.

□

(III) Suppose that G, H are groups and $\psi: G \to H$ is a homomorphism.

(a) Let's write $\psi(G) = \{\psi(g) \mid g \in G\}$. Prove that $\psi(G)$ is a subgroup of H.

(b) For J a subgroup of H, let's write $\psi^{-1}(J) = \{g \in G \mid \psi(g) \in J\}$. Prove that $\psi^{-1}(J)$ is a subgroup of G.

Proof. Let's write e_G, e_H for the identities of G, H, respectively.

(a) We must show three things.

- From class we know that $\psi(e_G) = e_H$, so that $e_H \in \psi(G)$ and $\psi(G) \neq \varnothing$.
- Suppose that $h_1, h_2 \in \psi(G)$. Then there are $g_1, g_2 \in G$ with $\psi(g_1) = h_1$ and $\psi(g_2) = h_2$. Since ψ is a homomorphism, we know that $h_1 h_2 = \psi(g_1)\psi(g_2) = \psi(g_1 g_2) \in \psi(G)$.
- Finally, choose any $h \in \psi(G)$, so there is some $g \in G$ with $\psi(g) = h$. From class, we know that $h^{-1} = (\psi(g))^{-1} = \psi\left(g^{-1}\right) \in \psi(G)$.

(b) We must show three things.

- From class we know that $\psi(e_G) = e_H \in J$, so that $e_G \in \psi^{-1}(J)$ and $\psi^{-1}(J) \neq \varnothing$.
- Suppose that $g_1, g_2 \in \psi^{-1}(J)$. Since ψ is a homomorphism and J is a subgroup, we know that $\psi(g_1 g_2) = \psi(g_1)\psi(g_2) \in J$, so that $g_1 g_2 \in \psi^{-1}(J)$.
- Finally, choose any $g \in \psi^{-1}(J)$. From class, and since J is a subgroup, we know that $\psi\left(g^{-1}\right) = (\psi(g))^{-1} \in J$, so that $g^{-1} \in \psi^{-1}(J)$.

□