HW 7 Due: 29 May 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Let $G = \mathbb{Z}_{20}$ and $H = \langle 15 \rangle$. Write down all elements of all (right) cosets of H in G.
2. Let $G = D_4$ and $H = \langle R \rangle$. Write down all elements of all (right) cosets of H in G.
3. For any group G and element $g \in G$, define the homomorphism

$$\begin{aligned} e_{G,g}\colon \mathbb{Z} &\to G \\ n &\mapsto g^n. \end{aligned}$$

   (No need to prove that $e_{G,g}$ is a homomorphism.) Let's write $e_{G,g}(\mathbb{Z}) = \{e_{G,g}(n) \mid n \in \mathbb{Z}\}$. Enumerate elements of $e_{G,g}(\mathbb{Z})$ and find $\ker(e_{G,g})$ in the following situations:

   (a) $G = D_4, g = R$,

   (b) $G = \mathbb{Z}_{20}, g = 15$,

   (c) $G = D_4, g = F$.

   (d) $G = S_9, g = \mathrm{id}_{\{1,2,3,4,5,6,7,8,9\}}$.

Proofs

(I) Let G be a group, let H be a subgroup of G, and let $a, b \in G$. Prove

$$Ha = Hb \qquad \text{if and only if} \qquad ab^{-1} \in H.$$

(II) Suppose n is a positive odd integer, that G is a group of order $2n$, and that $a, b \in G$ have orders $2, n$, respectively. Prove that

$$G = \left\{ a^i b^j \mid i \in \{0,1\} \text{ and } j \in \{0, \dots, n-1\} \right\}.$$

(III) Suppose that G, H are groups and $\psi\colon G \to H$ is a homomorphism.

   (a) Let's write $\psi(G) = \{\psi(g) \mid g \in G\}$. Prove that $\psi(G)$ is a subgroup of H.

   (b) For J a subgroup of H, let's write $\psi^{-1}(J) = \{g \in G \mid \psi(g) \in J\}$. Prove that $\psi^{-1}(J)$ is a subgroup of G.