HW 6 Due: 22 May 2024

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. Write down all the orders of all the elements of D_4.
2. Write down all the orders of all the elements of $S_3 \times \mathbb{Z}_2$.
3. Let $f = (14563) \in S_9$. Write down all the orders of all the elements of $\langle f \rangle$.

Proofs

(I) Let G be a group and suppose $g, h \in G$ have finite order. Prove: if $gh = hg$, then $\operatorname{ord}(gh)$ is a divisor of $\operatorname{lcm}(\operatorname{ord}(g), \operatorname{ord}(h))$.

(II) Suppose that G is a finite cyclic group of order n, with generator $g \in G$. Let $j \in \mathbb{Z}_{>0}$. Prove: if there exist $a, b \in \mathbb{Z}$ with $an + bj = 1$, then $G = \langle g^j \rangle$. (You can use [Pin10, Chapter 10, Theorem 1].)

(III) Let $m \in \mathbb{Z}_{>0}$ and let J be any subgroup of $\mathbb{Z}_m$. Prove that if j is the smallest positive integer in J, then $J = \langle j \rangle$. (In particular: all subgroups of $\mathbb{Z}_m$ are cyclic.)

(IV) Suppose that G, H are groups and $\phi: G \to H$ is an isomorphism. Prove: for all $g \in G$,

$$\operatorname{ord}(g) = \operatorname{ord}(\phi(g)).$$

(V) Let $\mathcal{F}$ be the set of all functions with domain and codomain $\mathbb{R}$. Define $\sim$ on $\mathcal{F}$ by setting for all $f, g \in \mathcal{F}$:

$$f \sim g \qquad \text{if and only if} \qquad f(0) = g(0).$$

Prove that $\sim$ is an equivalence relation on $\mathcal{F}$.

References

[Pin10] Charles C. Pinter, *A book of abstract algebra*, Dover Publications, Inc., Mineola, NY, 2010, Reprint of the second (1990) edition [of MR0644983]. MR 2850284