HW 1

DUE ON 4 OCTOBER, 2023

1. Computations

(1) Let $A = \{a, b, c\}$.
   (a) Write down all functions from A to itself, until you have convinced yourself that you *could* write down all such functions. How many are there? (So the answer to this question is a number.) Here is an example of a function from A to itself:
$$a \mapsto b$$
$$b \mapsto c$$
$$c \mapsto a.$$
   (b) Write down all functions f from A to itself with the property that $f \circ f$ is the identity function on A.

(2) Let $B = \{0, 1, 2, 3\}$. Let C be the following set:
$$C = \{(a, b) \in B \times B \mid \text{both } a \text{ and } b \text{ are odd}\}.$$
Write down all elements of C.

2. Proofs

Here are two example problems, with solutions.

Example 2.1. For any integers a, b, define $a \star b = |ab|$. Prove $\star$ is an operation on $\mathbb{Z}$.

Solution. Choose $a, b \in \mathbb{Z}$. Note that if a and b have the same sign, then $a \star b = ab$, which is an integer, and if a and b have different signs, then $a \star b = -ab$, which is also an integer. Thus, we see that $\star$ satisfies the definition of operation. □

Here are some comments about the example.

- Literally everything in the solution is part of a complete sentence.
- Before I manipulate a, b at you, I tell you specifically what they are. (This is required when writing a proof, or I will think to myself "who are these people a and b? I don't remember being introduced to them".)
- I don't restate the definition of operation, because that is part of the set of common knowledge for this class (ie, you are writing these proofs *for me*, and I know all the definitions.)

The "easier" version of this type of question is when you must show a statement is false—you need only provide a counterexample. Often you can even skip introductions, because you will not need variables!

Example 2.2. For any nonzero integers a, b, define $a \bullet b = a/b$. Prove that $\bullet$ is not an operation on $\mathbb{Z}$.

Solution. Note that 1,2 are integers but $1 \bullet 2 = 1/2$, which is not an integer. Therefore, we see that $\bullet$ is not an operation on $\mathbb{Z}$. □

Here are some exercises. Fill in the blanks *with complete sentences.*

(I) For any integers a, b, define $a \star b = a - b$. Prove $\star$ is an operation on $\mathbb{Z}$.

Solution. Choose $a, b \in \mathbb{Z}$. Note that $a \star b = a - b$, which is also an integer. __________________. □

(II) For any positive integers a, b, define $a \bullet b = a - b$. Prove $\bullet$ is not an operation on $\mathbb{Z}_{>0}$.

Solution. __________________. Therefore, we see that $\bullet$ is not an operation on $\mathbb{Z}_{>0}$. □

(III) For any $a, b \in \mathbb{Q}_{\neq 0}$, define $a \odot b = a/b$, Then $\odot$ is an operation on $\mathbb{Q}_{\neq 0}$.

Solution. Choose any $a, b \in \mathbb{Q}_{\neq 0}$. __________________. We see that

$$a \odot b = \frac{c}{d} \odot \frac{e}{f} = \frac{\frac{c}{d}}{\frac{e}{f}} = \frac{ce}{df}.$$

Since c, d, e, f are nonzero *as we noted above*, we see that cd, df are nonzero, so that $\frac{ce}{df} \in \mathbb{Q}_{\neq 0}$. Therefore, we conclude that $\odot$ is an operation on $\mathbb{Q}_{\neq 0}$. □

(IV) Consider for a moment the operation of multiplication on the integers. Prove that this operation admits an identity.

Solution. __________________. Thus, we see that 1 is the identity element for multiplication on the integers. □

Finally, here is an opportunity to write a complete proof:

(i) In Exercise (2), we defined the set C. For any $(a, b), (c, d) \in C$, define $(a, b) \oplus (c, d) = (a + c, b + d)$. Prove that $\oplus$ is not an operation on C.

3. Solutions

(1) (a) There are 27 functions from A to itself:

$$\begin{array}{ccccccccc}
a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a & a \mapsto a \\
b \mapsto a & b \mapsto a & b \mapsto a & b \mapsto b & b \mapsto b & b \mapsto b & b \mapsto c & b \mapsto c & b \mapsto c \\
c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c
\end{array}$$

$$\begin{array}{ccccccccc}
a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b & a \mapsto b \\
b \mapsto a & b \mapsto a & b \mapsto a & b \mapsto b & b \mapsto b & b \mapsto b & b \mapsto c & b \mapsto c & b \mapsto c \\
c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c
\end{array}$$

$$\begin{array}{ccccccccc}
a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c & a \mapsto c \\
b \mapsto a & b \mapsto a & b \mapsto a & b \mapsto b & b \mapsto b & b \mapsto b & b \mapsto c & b \mapsto c & b \mapsto c \\
c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c & c \mapsto a & c \mapsto b & c \mapsto c
\end{array}$$

(b)

$$\begin{array}{cccc}
a \mapsto a & a \mapsto a & a \mapsto c & a \mapsto b \\
b \mapsto b & b \mapsto c & b \mapsto b & b \mapsto a \\
c \mapsto c & c \mapsto b & c \mapsto a & c \mapsto c
\end{array}$$

(2) $\{(1,1),(1,3),(3,1),(3,3)\}$

(I) "Thus, we see that $\star$ is an operation on $\mathbb{Z}$."

(II) "Note that $1 \bullet 3 = 1 - 3 = -2$ is not a positive integer."

(III) "Since $a, b \in \mathbb{Q}_{\neq 0}$, there exist $c, d, e, f \in \mathbb{Z}$, all of which are nonzero, such that $a = \frac{c}{d}$ and $b = \frac{e}{f}$."

(IV) "Recall that for any $n \in \mathbb{Z}$, we know $1 \cdot n = n \cdot 1 = n$."

(i) By Exercise (2), we know that $(1,1) \in C$, but $(1,1) \oplus (1,1) = (2,2) \notin C$. Thus, we see that $\oplus$ is not an operation on C.