Computations

1. For the following element f of S_9, do the following:
   (i) write f in disjoint cycle form,
   (ii) write f as a product of transpositions,
   (iii) state the parity of f, and
   (iv) write $f \circ (146) \circ (329)$ in disjoint cycle form.

$$\begin{aligned} f\colon \{1,2,3,4,5,6,7,8,9\} &\to \{1,2,3,4,5,6,7,8,9\} \\ 1 &\mapsto 1 \\ 2 &\mapsto 5 \\ 3 &\mapsto 4 \\ 4 &\mapsto 2 \\ 5 &\mapsto 3 \\ 6 &\mapsto 7 \\ 7 &\mapsto 8 \\ 8 &\mapsto 9 \\ 9 &\mapsto 6 \end{aligned}$$

Solution. (i) (2534)(6789),
(ii) (24)(23)(25)(69)(68)(67),
(iii) even, and
(iv) (126)(35)(4789).

□

2. Suppose G is a group with a subgroup H. For any $g \in G$, define $gH = \{gh \mid h \in H\}$ and $Hg = \{hg \mid h \in H\}$.
   In the following four parts, you must enumerate some elements of D_4. Write your elements in our "standard form"; that is, as either "id" or "$F^i R^j$" for some $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$.
   (a) Enumerate all elements in $\langle R \rangle F$.
   (b) Enumerate all elements in $F \langle R \rangle$.
   (c) Enumerate all elements in $\langle F \rangle R$.
   (d) Enumerate all elements in $R \langle F \rangle$.

Solution. (a) $\langle R \rangle F = \{F, RF, R^2F, R^3F\} = \{F, FR^3, FR^2, FR\}$.
(b) $F \langle R \rangle = \{F, FR, FR^2, FR^3\}$.
(c) $\langle F \rangle R = \{R, FR\}$.
(d) $R \langle F \rangle F = \{R, RF\} = \{R, FR^3\}$.

□

Proofs

(I) Suppose that G is a group with a subgroup H. Suppose that $g_1, g_2 \in G$ satisfy $g_1 g_2 \in H$. Prove:

$$\text{if } g_1 \in H, \text{ then } g_2 \in H.$$

Proof. Since H is a subgroup, it is closed under inverses, so the fact that $g_1 \in H$ tells us that $g_1^{-1} \in H$. The fact that H is also closed under the operation of G—along with the associativity of the operation on G—tells us that

$$g_2 = \left(g_1^{-1} g_1\right) g_2 = g_1^{-1} \left(g_1 g_2\right) \in H.$$

□

(II) Suppose that G, H are groups with identities e_G, e_H, respectively. Next, suppose that $f\colon G \to H$ is an isomorphism. Prove that $f\left(e_G\right) = e_H$.

Proof. Note that

$$\begin{aligned} e_G f\left(e_G\right) &= f\left(e_G\right) && \text{(since } e_H \text{ is the identity of } G\text{)} \\ &= f\left(e_G e_G\right) && \text{(since } e_G \text{ is the identity of } G\text{)} \\ &= f\left(e_G\right) f\left(e_G\right) && \text{(since } f \text{ is an isomorphism)}, \end{aligned}$$

so we conclude by right cancellation. □

(III) Suppose that G is a group and H is a subgroup of G. Choose any $g \in G$, and let's write

$$gHg^{-1} = \left\{ghg^{-1} \mid h \in H\right\}.$$

Recall that in HW3, we proved that gHg^{-1} is a subgroup of G. (You do not need to prove this again here.) Prove that H is isomorphic to gHg^{-1}.

Proof. Let's define

$$\begin{aligned} \phi\colon H &\to gHg^{-1} \\ h &\mapsto ghg^{-1}. \end{aligned}$$

- To see that ϕ is injective, choose any $h_1, h_2 \in H$ and assume that $\phi\left(h_1\right) = \phi\left(h_2\right)$. Then

$$gh_1g^{-1} = \phi\left(h_1\right) = \phi\left(h_2\right) = gh_2g^{-1},$$

  so we see that $h_1 = h_2$ by right and left cancellation. Thus, we see that ϕ is injective.
- Choose any $h \in H$, so that $ghg^{-1} \in gHg^{-1}$ is arbitrary. Then $\phi(h) = ghg^{-1}$, so we see ϕ is surjective.
- Finally, choose any $h_1, h_2 \in H$ and note that

$$\phi\left(h_1\right)\phi\left(h_2\right) = \left(gh_1g^{-1}\right)\left(gh_2g^{-1}\right) = gh_1\left(g^{-1}g\right)h_2g^{-1} = g\left(h_1h_2\right)g^{-1} = \phi\left(h_1h_2\right),$$

  so we conclude that ϕ is an isomorphism.

□

Extra Credit (if you have extra time)

This exercise shows that *every* group is isomorphic to a subgroup of a symmetric group! Suppose that G is a group. For any $g \in G$, define

$$\begin{aligned} f_g \colon G &\to G \\ x &\mapsto gx. \end{aligned}$$

We have shown that $f_g \in S_G$. (You don't need to do this again.) Now define

$$\begin{aligned} \phi \colon G &\to S_G \\ g &\mapsto f_g. \end{aligned}$$

and let's write $\phi(G)$ for the set $\{\phi(g) \mid g \in G\}$, which is a subset of S_G.

(a) Prove that $\phi(G)$ is a subgroup of S_G.

(b) Prove that ϕ is injective.

(c) Prove that for all $g, h \in G$, we have that $\phi(gh) = \phi(g)\phi(h)$.

(d) Conclude that G is isomorphic to $\phi(G)$.

Proof. Let's write e for the identity of G.

(a) Note that for any $x \in G$,
$$f_{e_G}(x) = e_G x = x = \mathrm{id}_G(x),$$
so $\mathrm{id}_G \in \phi(G)$. Next, choose any $g_1, g_2 \in G$, so that $\phi(g_1), \phi(g_2)$ are arbitrary in $\phi(G)$. Then for any $x \in G$, we see that
$$\phi(g_1 g_2)(x) = f_{g_1 g_2}(x) = (g_1 g_2)(x) = g_1(g_2 x) = g_1(f_{g_2}(x)) = (f_{g_1} \circ f_{g_2})(x) = (\phi(g_1)\phi(g_2))(x);$$
that is, we see $\phi(g_1)\phi(g_2) = \phi(g_1 g_2) \in \phi(G)$, so that $\phi(G)$ is closed under the operation of S_G. Finally, choose any $g \in G$, so that $\phi(g)$ arbitrary in $\phi(G)$. Note that for any $x \in G$,
$$\left(\phi\left(g^{-1}\right)\phi(g)\right)(x) = \left(f_{g^{-1}} \circ f_g\right)(x) = g^{-1}(gx) = x = \mathrm{id}_G(x),$$
so we see that $(\phi(g))^{-1} = \phi\left(g^{-1}\right) \in \phi(G)$ by a fact from class (that is, in a group, we only need to check "one side" of inverses).

(b) Choose $g_1, g_2 \in G$, and assume $\phi(g_1) = \phi(g_2)$. Then
$$g_1 = g_1 e = f_{g_1}(e) = \phi(g_1)(e) = \phi(g_2)(e) = f_{g_2}(e) = g_2 e = g_2.$$

(c) Choose any $x \in G$ and note that
$$\phi(gh)(x) = f_{gh}(x) = (gh)x = g(hx) = g(f_h(x)) = f_g(f_h(x)) = (f_g \circ f_h)(x) = (\phi(g)\phi(h))(x),$$
so that $\phi(gh) = \phi(g)\phi(h)$.

(d) Let's define
$$\begin{aligned} \psi \colon G &\to \phi(G) \\ g &\mapsto \phi(g). \end{aligned}$$
By (a), we know that $\phi(G)$ is a group; by (c), we know ψ respects the operation of G; and by (b) and the definition of $\phi(G)$, we know that ψ is bijective.

□