Math 344 Spring 2024 Midterm

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. For the following element f of S_9, do the following:
   (i) write f in disjoint cycle form,
   (ii) write f as a product of transpositions,
   (iii) state the parity of f, and
   (iv) write $f \circ (146) \circ (329)$ in disjoint cycle form.

$$
\begin{aligned}
f\colon \{1,2,3,4,5,6,7,8,9\} &\to \{1,2,3,4,5,6,7,8,9\} \\
1 &\mapsto 1 \\
2 &\mapsto 5 \\
3 &\mapsto 4 \\
4 &\mapsto 2 \\
5 &\mapsto 3 \\
6 &\mapsto 7 \\
7 &\mapsto 8 \\
8 &\mapsto 9 \\
9 &\mapsto 6
\end{aligned}
$$

2. Suppose G is a group with a subgroup H. For any $g \in G$, define $gH = \{gh \mid h \in H\}$ and $Hg = \{hg \mid h \in H\}$.

   In the following four parts, you must enumerate some elements of D_4. Write your elements in our "standard form"; that is, as either "id" or "$F^i R^j$" for some $i \in \{0, 1\}$ and $j \in \{0, 1, 2, 3\}$.

   (a) Enumerate all elements in $\langle R \rangle F$.

   (b) Enumerate all elements in $F \langle R \rangle$.

   (c) Enumerate all elements in $\langle F \rangle R$.

   (d) Enumerate all elements in $R \langle F \rangle$.

Proofs

(I) Suppose that G is a group with a subgroup H. Suppose that $g_1, g_2 \in G$ satisfy $g_1 g_2 \in H$. Prove:

$$\text{if } g_1 \in H, \text{ then } g_2 \in H.$$

(II) Suppose that G, H are groups with identities e_G, e_H, respectively. Next, suppose that $f\colon G \to H$ is an isomorphism. Prove that $f(e_G) = e_H$.

(III) Suppose that G is a group and H is a subgroup of G. Choose any $g \in G$, and let's write

$$gHg^{-1} = \left\{ ghg^{-1} \mid h \in H \right\}.$$

Recall that in HW3, we proved that gHg^{-1} is a subgroup of G. (You do not need to prove this again here.) Prove that H is isomorphic to gHg^{-1}.

Extra Credit (if you have extra time)

This exercise shows that *every* group is isomorphic to a subgroup of a symmetric group! Suppose that G is a group. For any $g \in G$, define

$$\begin{aligned} f_g \colon G &\to G \\ x &\mapsto gx. \end{aligned}$$

We have shown that $f_g \in S_G$. (You don't need to do this again.) Now define

$$\begin{aligned} \phi \colon G &\to S_G \\ g &\mapsto f_g. \end{aligned}$$

and let's write $\phi(G)$ for the set $\{\phi(g) \mid g \in G\}$, which is a subset of S_G.

(a) Prove that $\phi(G)$ is a subgroup of S_G.

(b) Prove that ϕ is injective.

(c) Prove that for all $g, h \in G$, we have that $\phi(gh) = \phi(g)\phi(h)$.

(d) Conclude that G is isomorphic to $\phi(G)$.