Math 344 Fall 2023 Midterm

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. For the following element f of S_9, do the following:
   (i) write f in disjoint cycle form,
   (ii) write f as a product of transpositions,
   (iii) state the parity of f, and
   (iv) write $(126) \circ (389) \circ f$ in disjoint cycle form.

$$
\begin{aligned}
f\colon \{1,2,3,4,5,6,7,8,9\} &\to \{1,2,3,4,5,6,7,8,9\} \\
1 &\mapsto 7 \\
2 &\mapsto 2 \\
3 &\mapsto 4 \\
4 &\mapsto 5 \\
5 &\mapsto 3 \\
6 &\mapsto 1 \\
7 &\mapsto 8 \\
8 &\mapsto 9 \\
9 &\mapsto 6
\end{aligned}
$$

Solution. (i) (17896)(345),
(ii) (17)(78)(89)(96)(34)(45),
(iii) even,
(iv) (179)(26)(3458).

□

2. Suppose that G is a group with a subgroup H. For any $g \in G$, define $gH = \{gh \mid h \in H\}$. In disjoint cycle form, enumerate all elements of gH in the following situations:
   (a) $G = S_3$, $H = \langle (123) \rangle$, $g = (132)$,
   (b) $G = S_3$, $H = \langle (123) \rangle$, $g = (12)$, and
   (c) $G = S_3$, $H = \langle (123) \rangle$, $g = (23)$.

Solution. 1. $\{\mathrm{id}_{\{1,2,3\}}, (123), (132)\}$,

2. $\{(12), (13), (23)\}$, and

3. $\{(12), (13), (23)\}$.

□

Proofs

(I) Suppose that G is a group and let $D = \{(g,g) \mid g \in G\}$.

(a) Prove that D is a subgroup of $G \times G$.

(b) Prove that G is isomorphic to D.

Proof. Let e be the identity of G

(a) Since $(e,e) \in D$, we know $D \neq \varnothing$. Chose any $g, h \in G$, so that $(g,g),(h,h)$ are arbitrary in D. Then $(g,g)(h,h) = (gh, gh) \in D$, so D is closed under the operation of $G \times G$. Finally, choose any $g \in G$, so that (g,g) is arbitrary in D. Since $(g,g)\left(g^{-1}, g^{-1}\right) = \left(gg^{-1}, gg^{-1}\right) = (e,e)$, and (e,e) is the identity of $G \times G$ from previous work, we know by a class lemma that $(g,g)^{-1} = \left(g^{-1}, g^{-1}\right) \in D$, so D is closed under inverses. We conclude that D is a subgroup of $G \times G$ by our subgroup test.

(b) Define

$$\begin{aligned} \phi : G &\to D \\ g &\mapsto (g,g). \end{aligned}$$

To show ϕ is surjective, choose any $(g,g) \in D$ and note that $\phi(g) = (g,g)$. To show that ϕ is injective, choose any $g, h \in G$ and suppose that $\phi(g) = \phi(h)$. This tells us that $(g,g) = (h,h)$, so that $g = h$. Finally, to show that ϕ respects the operations of G and D, choose any $g, h \in G$ and use the definition of ϕ and the operation on $G \times G$ to note that

$$\phi(gh) = (gh, gh) = (g,g)(h,h) = \phi(g)\phi(h),$$

so we conclude that ϕ is an isomorphism.

□

(II) Suppose that G, H are groups and that $\phi : G \to H$ is an isomorphism. Prove: if G is commutative, then H is commutative.

Proof. Choose any $h_1, h_2 \in H$. Since ϕ is an isomorphism, it is surjective, so there are $g_1, g_2 \in G$ such that $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$. Using the assumption that G is commutative and the fact that ϕ respects the operations of G, H, we see

$$h_1 h_2 = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1) = h_2 h_1.$$

That is, we see that H is commutative. □

(III) Suppose that G is a group with identity e, and choose some $g \in G$. Define the function

$$\begin{aligned} f_g : G &\to G \\ x &\mapsto gx. \end{aligned}$$

(a) Prove: f_g is bijective.

(b) Prove: if f_g is an isomorphism, then $g = e$.

Proof. (a) To show that f_g is surjective, choose any $y \in G$ and note that $f_g\left(g^{-1}y\right) = g\left(g^{-1}\right)y = y$. To show that f_g is injective, suppose that $x_1, x_2 \in G$ and $f_g(x_1) = f_g(x_2)$, so that $gx_1 = gx_2$. But then we see that $x_1 = x_2$ by left cancellation.

(b) If f_g is an isomorphism, we know from class that $f_g(e) = e$. Then $g = ge = f_g(e) = e$, as desired. If we don't remember that fact, we could use the definitions of isomorphism and f to note that

$$ef_g(e) = f_g(e) = f_g\left(e^2\right) = f_g(e)f_g(e) = (ge)f_g(e) = gf_g(e),$$

then conclude that $e = g$ by right cancellation.

□

Extra Credit (if you have extra time)

Suppose that G is a group with two subgroups I and J. Prove that

$$I \cup J \text{ is a subgroup of } G \qquad \text{if and only if} \qquad \text{either } I \subseteq J \text{ or } J \subseteq I.$$

Proof.

- If $I \subseteq J$, then $I \cup J = J$ is a subgroup of G by hypothesis. Likewise, if $J \subseteq I$, then $I \cup J = I$ is a subgroup of G by hypothesis.
- Conversely, suppose that $I \cup J$ is a subgroup of G. If $I \subseteq J$, we are done! So we suppose that $I \nsubseteq J$; that is, there is some $i \in I$ with $i \notin J$. We must show that $J \subseteq I$, so take an arbitrary $j \in J$. Since $I \cup J$ is a subgroup, we know that $ij \in I \cup J$. There are two cases:
  - Suppose $ij \in J$. Since J is a subgroup, we know $j^{-1} \in J$. Closure of J now implies $i = (ij)j^{-1} \in J$—but this is not the case by our choice of i, so we see $ij \notin J$.
  - Suppose $ij \in I$. Since I is a subgroup, we know $i^{-1} \in I$. Closure of I now implies $j = i^{-1}(ij) \in I$, as desired.

□