Math 344 Spring 2020 Final

Due by 3:00am on Friday, June 12, 2020

Name:____

- Put your name in the "____" above.
- The only resources you should be using are
  - your book [Pin10],
  - your notes,
  - videos of lecture, and
  - talking to me (Derek) in class on Monday.
- Answer all problems.
- Good luck!

Proof questions

1. Suppose that G is a group of size 77. Suppose that H is a nontrivial normal subgroup of G. (In other words, suppose that $|H| > 1$.) Prove that G/H is cyclic.

   Proof. We will first show: for all groups J, if $|J| = p$ is prime, then J is cyclic. To see that this is true, we first use the fact that $p > 1$ to choose some nonidentity element j of J. By Lagrange's Theorem, we know that $\operatorname{ord}(j)$ divides p. But since j is not the identity element of J, we know that $\operatorname{ord}(j) \neq 1$. Since p is prime, the only other choice is $\operatorname{ord}(j) = p$, which means that J is cyclic with generator j.

   Proceeding to the problem, we know by Lagrange's Theorem that $|H|$ is a divisor of 77, so $|H| \in \{1, 7, 11, 77\}$. Since H is nontrivial, we see $|H| \in \{7, 11, 77\}$. Since all cosets of H are the same size, we see that $|G/H| \in \{11, 7, 1\}$. If $|G/H| \in \{11, 7\}$, we know by the first paragraph that G/H is cyclic. On the other hand, if $|G/H| = 1$, then G/H is generated by its identity element—namely, the coset H. □

2. Let $\phi: \mathbb{Z}_{100} \to \mathbb{Z}$ by any homomorphism of groups.

   (a) Prove that $\phi(1) = 0$.

   (b) What is $\ker \phi$? Prove your answer is correct.

   Proof. (a) Since ϕ is a homomorphism,

   $$0 = \phi(\overbrace{1 + \cdots + 1}^{100 \text{ times}}) = \overbrace{\phi(1) + \cdots + \phi(1)}^{100 \text{ times}} = 100 \cdot \phi(1).$$

   But the only integer n with the property that $100n = 0$ is $n = 0$.

   (b) Let $m \in \mathbb{Z}_{100}$ be nonzero. Then

   $$\phi(m) = \phi(\overbrace{1 + \cdots + 1}^{m \text{ times}}) = \overbrace{\phi(1) + \cdots + \phi(1)}^{m \text{ times}} = 100 \cdot 0 = 0.$$

   Of course, we also know that $\phi(0) = 0$, so we conclude that $\ker(\phi) = \mathbb{Z}_{100}$.

   □

3. Suppose that G is a group and $|G| < 200$. Suppose that G has subgroups of size 15 and 36. What is the size of G? Prove your answer is correct.

Proof. By Lagrange's Theorem, we know that 15 and 36 both divide $|G|$. But the only integer between 1 and 200 divisible by both 15 and 36 is 180. □

Computational questions

1. Using cycle notation, write down all elements of S_4 with order 2.

   Solution. $(12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)$. □

2. Using cycle notation, write down all elements of S_4 with order 4.

   Solution. $(1234), (1243), (1324), (1342), (1423), (1432)$. □

3. (a) Using cycle notation, write down one element of S_4 with order 3, and let's call it f. (And let's agree to write "id" for the identity element of S_4.)
   (b) Using cycle notation, write down all elements of $\langle f \rangle$.
   (c) What is $|S_4/\langle f \rangle|$?
   (d) Write down all elements of $S_4/\langle f \rangle$.

   Solution. (a) $f = (123)$.
   (b) $\mathrm{id}, (123), (132)$.
   (c) $24/3 = 8$.
   (d)
   - $\langle f \rangle = \{\mathrm{id}, (123), (132)\}$,
   - $\langle f \rangle (12) = \{(12), (13), (23)\}$,
   - $\langle f \rangle (14) = \{(14), (1423), (1432)\}$,
   - $\langle f \rangle (24) = \{(24), (1243), (1324)\}$,
   - $\langle f \rangle (34) = \{(34), (1234), (1342)\}$,
   - $\langle f \rangle (12)(34) = \{(12)(34), (134), (234)\}$,
   - $\langle f \rangle (13)(24) = \{(13)(24), (243), (124)\}$,
   - $\langle f \rangle (14)(23) = \{(14)(23), (142), (143)\}$.

   □

References

[Pin10] Charles C. Pinter, *A book of abstract algebra*, Dover Publications, Inc., Mineola, NY, 2010, Reprint of the second (1990) edition [of MR0644983]. MR 2850284