

Exam One

Math 344

- Write your name here: _____.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

Computations

1. Consider the following two functions, to which we will give the names g and h for this problem:

$$\begin{array}{ll} g: \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\} & h: \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ \begin{array}{l} 1 \mapsto 3 \\ 2 \mapsto 2 \\ 3 \mapsto 4 \\ 4 \mapsto 6 \\ 5 \mapsto 7 \\ 6 \mapsto 1 \\ 7 \mapsto 9 \\ 8 \mapsto 1 \\ 9 \mapsto 5 \end{array} & \begin{array}{l} 1 \mapsto 3 \\ 2 \mapsto 6 \\ 3 \mapsto 3 \\ 4 \mapsto 3 \\ 5 \mapsto 8 \\ 6 \mapsto 3 \\ 7 \mapsto 5 \\ 8 \mapsto 1 \\ 9 \mapsto 9. \end{array} \end{array}$$

- (a) Compute $g \circ h(1)$,
(b) compute $h \circ g(1)$,
(c) write the smallest positive integer n such that

$$\overbrace{g \circ \cdots \circ g}^{n \text{ times}}(1) = 1,$$

and

- (d) write down an element m in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with the property that $h \circ h(m) = h(m)$.

Solution.

- (a) 4,
(b) 3,
(c) 4,
(d) any of 1, 3, 4, 6, or 9.

□

2. (a) Write down three subgroups of D_6 that have size 2.
(b) Write down all elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ that are their own inverse.
(c) Write down three subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_6$ of size 3.
(d) Write down a group that has at least one subgroup of size 2 and at least one subgroup of size 7.

Solution.

- (a) There are, for example: $\langle R^3 \rangle, \langle F \rangle, \langle FR \rangle, \langle FR^2 \rangle, \langle FR^3 \rangle$.
(b) These elements are $(0, 0), (0, 2), (2, 0), (2, 2)$.
(c) For example: $\langle (2, 0) \rangle, \langle (2, 2) \rangle, \langle (0, 2) \rangle, \langle (2, 4) \rangle$.
(d) For example: $\mathbb{Z}_{14}$ and D_7 .

□

Proofs

1. Let $H = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m + 3n = 0\}$. Prove that H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

Proof. We will apply the subgroup test.

- Since $2(0) + 3(0) = 0$, we see $(0, 0) \in H$, so $H \neq \emptyset$.
- Suppose that $(a, b), (c, d) \in H$, so that $2a + 3b = 0 = 2c + 3d$. By the definition of the operation of $\mathbb{Z} \times \mathbb{Z}$, we know that

$$(a, b) + (c, d) = (a + c, b + d).$$

Since $2a + 3b = 0 = 2c + 3d$, we see that

$$2(a + c) + 3(b + d) = (2a + 3b) + (2c + 3d) = 0 + 0 = 0,$$

so $(a, b) + (c, d) \in H$ by the definition of H .

- Suppose that $(e, f) \in H$, so that $2e + 3f = 0$. From class, we know the $-(e, f) = (-e, -f)$. Using the fact that $2e + 3f = 0$, we see that $2(-e) + 3(-f) = -(2e + 3f) = -0 = 0$, so $-(e, f) \in H$.

Thus, we conclude that H is a subgroup by the subgroup test. □

2. Suppose that G is a group with a subgroup H . Suppose that $g_1, g_2 \in G$ satisfy $g_1 g_2 \in H$. Prove:

if $g_1 \in H$, then $g_2 \in H$.

Proof. Since H is a subgroup and $g_1 \in H$, we know that $(g_1)^{-1} \in H$. Since H is a subgroup, it is closed under the operation of G , so H contains

$$(g_1)^{-1} (g_1 g_2) = ((g_1)^{-1} g_1) g_2 = g_2.$$

□

3. Define the operation $\star$ on $\mathbb{Z}_{\geq 3}$ by

$$m \star n = \begin{cases} m & \text{if } m \geq n \\ n & \text{if } n \geq m. \end{cases}$$

- (a) Prove that $\star$ admits an identity element.
- (b) Either prove that $(\mathbb{Z}_{\geq 3}, \star)$ is a group or prove that it is not a group.

Proof.

- (a) For any $n \in \mathbb{Z}_{\geq 3}$, we know $3 \geq n$; thus, $3 \star n = n = n \star 3$, so 3 is an identity element for $\star$.
- (b) For any $n \in \mathbb{Z}_{\geq 3}$, we know $4 \star n \geq 4$. So 4 does not have an inverse, and $(\mathbb{Z}_{\geq 3}, \star)$ is not a group.

□