HW 3 Due: 20 October 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1 Computations

1. Write down every subgroup of $\mathbb{Z}_5$. (You can use "generator" notation. For example, $\langle 1 \rangle = \{0, 1, 2, 3, 4\}$.)
2. Write down every subgroup of $\mathbb{Z}_{10}$.
3. Write down every subgroup of $\mathbb{Z}_{70}$.
4. Do you have a conjecture about the number of subgroups of cyclic groups? (No need to turn in your answer to this question.)
5. How many surjective functions are there from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2$? How many injective functions?

2 Proofs

(I) Let G, H be groups with identities, e_G, e_H, respectively. Prove that $\{(e_G, h) \mid h \in H\}$ is a subgroup of $G \times H$. (A similar proof shows that $\{(g, e_H) \mid g \in G\}$ is a subgroup of $G \times H$, but you don't need to write this up.)

(II) Let G be a group, and define

$$C = \{g \in G \mid \text{for all } x \in G,\ xg = gx\}.$$

Prove that C is a subgroup of G.

(III) Let G be a group, let H be a subgroup of G, and choose any $g \in G$. Let's use the notation

$$gHg^{-1} = \left\{ghg^{-1} \mid h \in H\right\}$$

Prove gHg^{-1} is a subgroup of G.

(IV) Let A be a set and $a \in A$. Define

$$G = \{f \in S_A \mid f(a) = a\}.$$

Prove that G is a subgroup of S_A.