Math 261 Spring 2025 Exam Two

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Solutions are graded for correctness, clarity, rigor, neatness.
- Good luck!

1. For a a real number, let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -3 & -4 \\ 1 & 2 & a+5 & 4 \\ 2 & 4 & 6 & a+6 \end{bmatrix}.$$

(a) Compute $\det(A)$. (Hint: please remember that row operations can make computing determinants *much easier*.)

(b) For which value(s) of a is A invertible?

Solution. Note that

$$\det A = \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -3 & -4 \\ 1 & 2 & a+5 & 4 \\ 2 & 4 & 6 & a+6 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & -3 & -4 \\ 0 & 1 & a+2 & 0 \\ 0 & 2 & 0 & a-2 \end{bmatrix} = (-1)\det \begin{bmatrix} -1 & -3 & -4 \\ 0 & a+2 & 0 \\ 0 & 0 & a-2 \end{bmatrix} = (-1)(-1)\det \begin{bmatrix} a+2 & 0 \\ 0 & a-2 \end{bmatrix} = (a+2)(a-2),$$

so A is not invertible if $a = \pm 2$ and A is invertible if $a \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. □

2. Let

- L_1 be the line through the points $(1,2,3)$ and $(5,6,7)$ and
- L_2 be the line between $(1,6,2)$ and $(-7,2,-7)$.

(a) Write down a direction vector for L_1 and a vector it contains. (These are the constituents of the "vector equation" for L_1.)

(b) Write down a direction vector for L_2 and a vector it contains. (These are the constituents of the "vector equation" for L_2.)

(c) Do these lines intersect? If so, where do they intersect?

Solution. (a) $\mathbf{d}_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) $\mathbf{d}_2 = \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

(c) We row reduce:

$$\left[\begin{array}{cc|c} 4 & 8 & 0 \\ 4 & 4 & 4 \\ 4 & 9 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -4 & -4 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

so the lines intersect at

$$2\mathbf{d}_1 + \mathbf{v}_1 = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix} = (-1)\mathbf{d}_2 + \mathbf{v}_2.$$

□

3. Suppose that b be a real number and let

$$\mathbf{u} = \begin{bmatrix} b \\ -2b+4 \\ 2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

For which value(s) of b is $\mathbf{u}$ in Span $\{\mathbf{v}, \mathbf{w}\}$?

Solution. We must solve the equation

$$x\mathbf{v} + y\mathbf{w} = \mathbf{u},$$

so we row reduce:

$$\begin{bmatrix} 1 & 2 & b \\ 2 & 2 & -2b+4 \\ 3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & b \\ 0 & -2 & -4b+4 \\ 0 & -4 & -3b+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & b \\ 0 & -2 & -4b+4 \\ 0 & 0 & 5b-6 \end{bmatrix}.$$

This system is solvable precisely when $5b - 6 = 0$, so

- when $b = \frac{6}{5}$, the vector $\mathbf{u}$ is in Span $\{\mathbf{v}, \mathbf{w}\}$ and
- when $b \neq \frac{6}{5}$, the vector $\mathbf{u}$ is not in Span $\{\mathbf{v}, \mathbf{w}\}$.

□

4. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a vector $\mathbf{v}$ such that

- The first coordinate of $\mathbf{v}$ is 7 and
- $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

Solution. We must find $y, z \in \mathbb{R}$ such that

$$7 + 2y + z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} = 0.$$

One possible choice is $y = 0$ and $z = -7$, yielding the vector

$$\mathbf{v} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}.$$

□

5. (a) Write down a line and a plane in $\mathbb{R}^3$ that do not intersect.
(b) Write down two lines in $\mathbb{R}^3$ that do not intersect.
(c) Write down two lines in $\mathbb{R}^3$ that intersect exactly once.

Solution. (a) One example is $z = 0$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}t + \begin{bmatrix}0\\0\\1\end{bmatrix}$.

(b) One example is $\begin{bmatrix}1\\0\\0\end{bmatrix}t + \begin{bmatrix}0\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}t + \begin{bmatrix}0\\0\\0\end{bmatrix}$.

(c) One example is $\begin{bmatrix}1\\0\\0\end{bmatrix}t + \begin{bmatrix}0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}t + \begin{bmatrix}0\\0\\0\end{bmatrix}$.

□

6. Let

$$B = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

(a) Find all the eigenvalues of B.

(b) For each eigenvalue of B, find an associated eigenvector of B.

Solution. (a) Since $\det(\lambda I - B) = (\lambda-1)(\lambda-2)-12 = \lambda^2-3\lambda-10 = (\lambda+2)(\lambda-5)$, we know B has exactly two eigenvalues: -2 and 5.

(b) Since

$$-2I - B = \begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix},$$

one eigenvector for $\lambda = -2$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. And since

$$5I - B = \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix},$$

one eigenvector for $\lambda = 5$ is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

□

7. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find a real number c such that

$$\operatorname{proj}_{\mathbf{u}}(c\mathbf{v}) = 20\mathbf{u}.$$

Solution. Note that

$$\operatorname{proj}_{\mathbf{u}}(c\mathbf{v}) = \frac{c\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{6c}{3}\mathbf{u} = 2c\mathbf{u},$$

so $c = 10$ works. □

Extra credit

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a vector $\mathbf{v}$ such that

- The first coordinate of $\mathbf{v}$ is 7,
- $\mathbf{v}$ is perpendicular to $\mathbf{u}$, and
- $\|\mathbf{v}\| = \sqrt{83}$.

Solution. We must find $y, z \in \mathbb{R}$ such that

$$7 + 2y + z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \sqrt{49 + y^2 + z^2} = \left\| \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} \right\| = \sqrt{83}.$$

These equations imply

$$z = -7 - 2y \quad \text{and} \quad 49 + y^2 + z^2 = 83.$$

Substituting, we obtain

$$49 + y^2 + (-7 - 2y)^2 = 83,$$

which simplifies to

$$0 = 5y^2 + 28y + 15 = (5y + 3)(y + 5),$$

so the only possibilities are $y = -5$ and $y = -3/5$. Since $z = -7 - 2y$, we obtain two candidates for $\mathbf{v}$:

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ -5 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ -3/5 \\ -29/5 \end{bmatrix}.$$

By direct computation we see that

$$\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{83} \quad \text{and} \quad \mathbf{v}_1 \cdot \mathbf{u} = \mathbf{v}_2 \cdot \mathbf{u} = 0,$$

as desired. $\square$