Math 261 Spring 2025 Exam One

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Solutions are graded for correctness, clarity, rigor, neatness.
- Good luck!

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}.$$

Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find all solutions to the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

Solution. We solve by row reducing:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 5 \\ 7 & 8 & 9 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the set of solutions to the matrix equation is

$$\left\{ \begin{bmatrix} t \\ 1 - 2t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

$\square$

2. Suppose that

   - B is a 2×3 matrix,
   - C is a 3×2 matrix,
   - D is a 3×3 matrix, and
   - E is a 2×1 matrix.

   For each of the following matrix expressions, either tell me the size of the matrix or write "Undefined."

   (a) BC

   Solution. This is a 2×2 matrix. □

   (b) CB

   Solution. This is a 3×3 matrix. □

   (c) $B + C$

   Solution. Undefined. □

   (d) $C + B$

   Solution. Undefined. □

   (e) BD

   Solution. This is a 2×3 matrix. □

   (f) BE

   Solution. Undefined. □

   (g) BDC

   Solution. This is a 2×2 matrix. □

3. Define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\mathbf{x})$ be the vector obtained by rotating $\mathbf{x}$ counterclockwise by 270° (which is the same as $\frac{3\pi}{2}$ radians).

(a) Compute

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

Solution. Visually, we see

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

□

(b) You may assume that T is a linear transformation. Find a matrix F such that for all $\mathbf{x} \in \mathbb{R}^2$,

$$T(\mathbf{x}) = F\mathbf{x}.$$

Solution. Since

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we see

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

□

4. Let

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find G^{-1}, if it exists.

Proof. We solve by row reducing:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -6 & -4 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -7 & -6 & -4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -1 & 9 \\ 0 & 1 & 0 & -2 & -1 & 6 \\ 0 & 0 & 1 & 3 & 1 & -7 \end{array}\right],$$

so

$$G^{-1} = \begin{bmatrix} -4 & -1 & 9 \\ -2 & -1 & 6 \\ 3 & 1 & -7 \end{bmatrix}.$$

□

5. (a) Write a system of two linear equations in two variables that has infinitely many solutions.

Solution. One example is the system

$$\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0. \end{aligned}$$

□

(b) Solve your system from part (a).

Solution. We row reduce to see that the set of solutions to my system is

$$\left\{ \begin{bmatrix} t \\ -t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

□

6. Let

$$H = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) Let E_1 be the elementary matrix associated to scaling row two of H by 2. What is E_1?

Solution.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(b) Let E_2 be the elementary matrix associated to adding $(-1) \cdot$ (row one) to row two. What is E_2?

Solution.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(c) Let E_3 be the elementary matrix associated to adding $(-1) \cdot$ (row three) to row two. What is E_3?

Solution.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(d) What is $E_3E_2E_1H$?

Solution. We perform the row operations mentioned above to obtain

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{bmatrix}.$$

□

7. Suppose that a is a real number, and let

$$J = \begin{bmatrix} a & 2a \\ 3a & 4a+2 \end{bmatrix}.$$

For which values of a is J invertible? (Hint: first consider the case where $a = 0$, then consider all other cases.)

Solution. If $a = 0$, then the first column is all 0s, so there is leading one in the first column of $[G|I]$, and we see that G is not invertible in this case.

On the other hand, if $a \neq 0$, we solve by row reducing:

$$\left[\begin{array}{cc|cc} a & 2a & 1 & 0 \\ 3a & 4a+2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{a} & 0 \\ 3a & 4a+2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{a} & 0 \\ 0 & -2a+2 & -3 & 1 \end{array}\right].$$

So we see that if $a \neq 0$, the system is solvable precisely when $-2a + 2 \neq 0$—that is, when $a \neq 1$.

Thus,

- when $a \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$, the matrix G is invertible and
- when $a = 0, 1$, the matrix G is not invertible.

□

Extra credit

Let

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Write down a 9×9 matrix J such that

$$J\mathbf{x} = \mathbf{b} \text{ has infinitely many solutions} \qquad \text{and} \qquad J\mathbf{x} = \mathbf{b} \text{ has } \mathbf{x}_0 \text{ as a solution.}$$

Solution. One example is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□