Math 261 Spring 2025 Exam One

Name:_____

- Put your name in the "_____" above.
- Answer all questions.
- Solutions are graded for correctness, clarity, rigor, neatness.
- Good luck!

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}.$$

Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find all solutions to the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

2. Suppose that

   - B is a 2×3 matrix,
   - C is a 3×2 matrix,
   - D is a 3×3 matrix, and
   - E is a 2×1 matrix.

   For each of the following matrix expressions, either tell me the size of the matrix or write "Undefined."

   (a) BC

   (b) CB

   (c) $B + C$

   (d) $C + B$

   (e) BD

   (f) BE

   (g) BDC

3. Define a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\mathbf{x})$ be the vector obtained by rotating $\mathbf{x}$ counterclockwise by 270° (which is the same as $\frac{3\pi}{2}$ radians).

   (a) Compute

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

   (b) You may assume that T is a linear transformation. Find a matrix F such that for all $\mathbf{x} \in \mathbb{R}^2$,

$$T(\mathbf{x}) = F\mathbf{x}.$$

4. Let

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find G^{-1}, if it exists.

5. (a) Write a system of two linear equations in two variables that has infinitely many solutions.

(b) Solve your system from part (a).

6. Let

$$H = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) Let E_1 be the elementary matrix associated to scaling row two of H by 2. What is E_1?

(b) Let E_2 be the elementary matrix associated to adding $(-1) \cdot (\text{row one})$ to row two. What is E_2?

(c) Let E_3 be the elementary matrix associated to adding $(-1) \cdot (\text{row three})$ to row two. What is E_3?

(d) What is $E_3E_2E_1H$?

7. Suppose that a is a real number, and let

$$J = \begin{bmatrix} a & 2a \\ 3a & 4a+2 \end{bmatrix}.$$

For which values of a is J invertible? (Hint: first consider the case where $a = 0$, then consider all other cases.)

Extra credit

Let

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Write down a 9×9 matrix J such that

$$J\mathbf{x} = \mathbf{b} \text{ has infinitely many solutions} \quad \text{and} \quad J\mathbf{x} = \mathbf{b} \text{ has } \mathbf{x}_0 \text{ as a solution.}$$