Math 261 Fall 2024 Final Exam

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Let

$$A = \begin{bmatrix} 2 & -75 & 0 \\ 0 & -3 & 0 \\ 0 & -10 & 2 \end{bmatrix}.$$

(a) What are the eigenvalues of A and what are their algebraic multiplicities?

(b) For every eigenvalue found in part (a), find the dimension of its associated eigenspace. (In other words: find the geometric multiplicity of every eigenvalue of A).

(c) Is A diagonalizable? How do you know?

Solution. (a) We compute the characteristic polynomial of A:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 75 & 0 \\ 0 & \lambda + 3 & 0 \\ 0 & 10 & \lambda - 2 \end{bmatrix} = (\lambda - 2) \det \begin{bmatrix} \lambda + 3 & 0 \\ 10 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 (\lambda + 3),$$

so we see that the eigenvalues of A are $-3, 2$, with algebraic multiplicities $1, 2$, respectively.

(b)
- For $\lambda = -3$,

$$\left[\begin{array}{ccc|c} -5 & 75 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -5 & 0 \end{array}\right]$$

has one free variable, so the eigenvalue $\lambda = -3$ has geometric multiplicity 1.
- For $\lambda = 2$,

$$\left[\begin{array}{ccc|c} 0 & 75 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{array}\right]$$

has one two variables, so the eigenvalue $\lambda = 2$ has geometric multiplicity 2.

(c) Since the geometric multiplicities of the eigenvalues of A equal the algebraic multiplicities of the eigenvalues of A, we know that A is diagonalizable.

□

2. Give an example of:

   (a) a 3×2 matrix with a one-dimensional null space.
   (b) a 2×2 matrix that is invertible but not diagonalizable.
   (c) a 2×2 matrix that is diagonalizable but not invertible.
   (d) a 4×4 matrix with exactly two eigenvalues.
   (e) a 5×5 matrix with rank 2.
   (f) a 2×2 matrix with characteristic equation $\lambda^2 - 4\lambda + 3$.

   Proof. Some examples are

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$,

   (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

   (c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

   (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

   (e) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and

   (f) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

   □

3. Suppose that $k \in \mathbb{R}$ and let

$$B = \begin{bmatrix} k & k^2 & k^3 \\ -k & k - k^2 & 3 - k^3 \\ 0 & -2 & k - 5 \end{bmatrix}$$

For which value(s) of k does B have rank 3?

Solution. We compute:

$$\det \begin{bmatrix} k & k^2 & k^3 \\ -k & k - k^2 & 3 - k^3 \\ 0 & -2 & k - 5 \end{bmatrix} = \det \begin{bmatrix} k & k^2 & k^3 \\ 0 & k & 3 \\ 0 & -2 & k - 5 \end{bmatrix} = k \cdot \det \begin{bmatrix} k & 3 \\ -2 & k - 5 \end{bmatrix} = k\,(k(k-5)+6) = k(k-2)(k-3),$$

so B has rank 3 when $k \in (-\infty, 0) \cup (0, 2) \cup (2, 3) \cup (3, \infty)$. □

4. Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ by the rule

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x \\ 3x+y+z \\ 5x+3y+3z \end{bmatrix}.$$

Let C be the matrix for T.

(a) What is C?

(b) Find a basis for $\operatorname{im}(T)$.

Solution. (a) Since

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix},$$

we see that

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix}.$$

(b) We row reduce

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so a basis for $\operatorname{im}(T)$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}\right\}$.

□

5. Define the line

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

and for $a \in \mathbb{R}$, define the line

$$M = \left\{ \begin{bmatrix} a^2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

For which value(s) of a do the lines L and M intersect?

Solution. We row reduce

$$\left[\begin{array}{cc|c} 1 & -1 & a^2 - 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & a^2 - 1 \\ 0 & 1 & -a^2 \\ 0 & 0 & -a^2 + 1 \end{array}\right],$$

which has a solution exactly when $-a^2 + 1 = 0$; that is, when $a \in \{-1, 1\}$. □

6. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ -8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -9 \\ -14 \end{bmatrix}$$

and define

$$V = \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} \cdot \mathbf{u} = 0 \text{ and } \mathbf{x} \cdot \mathbf{v} = 0 \right\}.$$

Assuming V is a subspace of $\mathbb{R}^4$ (which it is), find a basis for V.

Solution. We row reduce

$$\left[\begin{array}{cccc|c} 2 & 1 & -5 & -8 & 0 \\ 3 & 2 & -9 & -14 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 1 & -5 & -8 & 0 \\ 1 & 1 & -4 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -4 & -6 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & -3 & -4 & 0 \end{array}\right],$$

which has basis of solutions $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

7. Find a formula in terms of k for the entries of A^k, where A is the diagonalizable matrix below and $P^{-1}AP = D$ for the matrices P and D below:

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Solution. First, we compute the inverse of P:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right].$$

Then we note that for any nonnegative integer k,

$$A^k = \left(PDP^{-1}\right)^k = PD^kP^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3^k & 2\cdot 3^k \\ (-1)^k 2^k & (-1)^{k+1}2^k \end{bmatrix} = \begin{bmatrix} -3^k + (-1)^k 2^{k+1} & 2\cdot 3^k + (-1)^{k+1}2^{k+1} \\ -3^k + (-1)^k 2^k & 2\cdot 3^k + (-1)^{k+1}2^k \end{bmatrix}.$$

□