Math 261 Fall 2024 Final Exam

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Let

$$A = \begin{bmatrix} 2 & -75 & 0 \\ 0 & -3 & 0 \\ 0 & -10 & 2 \end{bmatrix}.$$

   (a) What are the eigenvalues of A and what are their algebraic multiplicities?

   (b) For every eigenvalue found in part (a), find the dimension of its associated eigenspace. (In other words: find the geometric multiplicity of every eigenvalue of A).

   (c) Is A diagonalizable? How do you know?

2. Give an example of:

   (a) a 3×2 matrix with a one-dimensional null space.

   (b) a 2×2 matrix that is invertible but not diagonalizable.

   (c) a 2×2 matrix that is diagonalizable but not invertible.

   (d) a 4×4 matrix with exactly two eigenvalues.

   (e) a 5×5 matrix with rank 2.

   (f) a 2×2 matrix with characteristic equation $\lambda^2 - 4\lambda + 3$.

3. Suppose that $k \in \mathbb{R}$ and let

$$B = \begin{bmatrix} k & k^2 & k^3 \\ -k & k - k^2 & 3 - k^3 \\ 0 & -2 & k - 5 \end{bmatrix}$$

For which value(s) of k does B have rank 3?

4. Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ by the rule

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x \\ 3x+y+z \\ 5x+3y+3z \end{bmatrix}.$$

Let C be the matrix for T.

(a) What is C?

(b) Find a basis for $\operatorname{im}(T)$.

5. Define the line

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

and for $a \in \mathbb{R}$, define the line

$$M = \left\{ \begin{bmatrix} a^2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

For which value(s) of a do the lines L and M intersect?

6. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ -8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -9 \\ -14 \end{bmatrix}$$

and define

$$V = \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} \cdot \mathbf{u} = 0 \text{ and } \mathbf{x} \cdot \mathbf{v} = 0 \right\}.$$

Assuming V is a subspace of $\mathbb{R}^4$ (which it is), find a basis for V.

7. Find a formula in terms of k for the entries of A^k, where A is the diagonalizable matrix below and $P^{-1}AP = D$ for the matrices P and D below:

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$