Math 261 Section 6 Fall 2019 Exam 2 November 20, 2019

Name:____

- Put your name in the "______________________________________" above.
- Write your answers down neatly, use complete sentences, and *justify your work*.
- The extra credit question is for your amusement, if you have extra time at the end of the test.
- Good luck!

1. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

If possible, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution. To see if $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, we row reduce the appropriate system:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 3 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so we see that

$$\mathbf{w} = 2\mathbf{u} + 1\mathbf{v}.$$

□

2. Suppose that a is a real number, and let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -3 & -4 \\ 1 & 2 & a+5 & 4 \\ 2 & 4 & 6 & a+6 \end{pmatrix}.$$

For which value(s) of a is A invertible?

Solution. Note that

$$\det A = \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -3 & -4 \\ 1 & 2 & a+5 & 4 \\ 2 & 4 & 6 & a+6 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & -3 & -4 \\ 1 & 2 & a+5 & 4 \\ 2 & 4 & 6 & a+6 \end{pmatrix} = (-1) \det \begin{pmatrix} -1 & -3 & -4 \\ 1 & a+5 & 4 \\ 2 & 6 & a+6 \end{pmatrix}$$

$$= (-1) \det \begin{pmatrix} -1 & -3 & -4 \\ 0 & a+2 & 0 \\ 0 & 0 & a-2 \end{pmatrix}$$

$$= (-1)(-1) \det \begin{pmatrix} a+2 & 0 \\ 0 & a-2 \end{pmatrix}$$

$$= (a+2)(a-2),$$

so A is not invertible if $a = \pm 2$ and A is invertible if $a \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. □

3. Suppose that

- L_1 be the line through the points $(1,2,3)$ and $(5,6,7)$ and
- L_2 be the line between $(1,6,2)$ and $(-7,2,-7)$.

(a) What is the vector equation of L_1?
(b) What is the vector equation of L_2?
(c) Do these lines intersect? If so, where do they intersect?

Solution. (a) Let $\mathbf{d}_1$ be the vector connecting the two points in part (a), so that

$$\mathbf{d}_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix};$$

then L_1 is given by the vector equation

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}.$$

(b) Let $\mathbf{d}_2$ be the vector connecting the two points in part (b), so that

$$\mathbf{d}_2 = \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix};$$

then L_1 is given by the vector equation

$$\begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + t \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix}.$$

(c) Note that

$$\mathbf{d}_1 \cdot \mathbf{d}_2 =$$

(d) The lines intersect if they have a point in common; that is, if there are $s, t \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + t \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix}.$$

Thus, we must attempt to solve the system

$$\begin{aligned} 1 + 4s &= 1 - 8t \\ 2 + 4s &= 6 - 4t \\ 3 + 4s &= 2 - 9t, \end{aligned}$$

so we row reduce:

$$\begin{bmatrix} 4 & 8 & 0 \\ 4 & 4 & 4 \\ 4 & 9 & -1 \end{bmatrix} \sim \begin{bmatrix} 4 & 8 & 0 \\ 0 & -4 & 4 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 4 & 8 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so we find a point on L_1 and L_2:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix}.$$

□

4. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}.$$

Find two vectors (let's call them $\mathbf{u}_{\parallel}$ and $\mathbf{u}_{\perp}$) such that

- $\mathbf{u}_{\parallel}$ is a scalar multiple of $\mathbf{v}$,
- $\mathbf{u}_{\perp}$ is perpendicular to $\mathbf{v}$, and
- $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$.

HINT: this is a projection question—recall that $\text{proj}_{\mathbf{v}}(\mathbf{u})$ is a scalar multiple of $\mathbf{v}$.

Solution. We compute

$$\mathbf{u}_{\parallel} = \text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{2+8+4}{4+16+16} \mathbf{v} = \frac{7}{18} \mathbf{u} = \begin{bmatrix} 7/9 \\ 14/9 \\ 14/9 \end{bmatrix}.$$

Thus, we set

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 7/9 \\ 14/9 \\ 14/9 \end{bmatrix} = \begin{bmatrix} 2/9 \\ 4/9 \\ -5/9 \end{bmatrix}.$$

To check our work, note that

- $\mathbf{u}_{\perp} \cdot \mathbf{v} = \frac{4}{9} + \frac{16}{9} - \frac{20}{9} = 0$ and
- $\mathbf{u}_{\parallel} + \mathbf{u}_{\perp} = \begin{bmatrix} 7/9 + 2/9 \\ 14/9 + 4/9 \\ 14/9 + (-5/9) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$

□

5. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a vector $\mathbf{v}$ such that

- The first coordinate of $\mathbf{v}$ is 7 and
- $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

Proof. We must find $y, z \in \mathbb{R}$ such that

$$7 + 2y + z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} = 0.$$

One possible choice is $y = 0$ and $z = 7$, yielding the vector

$$\mathbf{v} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}.$$

□

6. Suppose that b be a real number and let

$$\mathbf{u} = \begin{bmatrix} b \\ -2b+4 \\ 2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

For which value(s) of b is $\mathbf{u}$ in Span $\{\mathbf{v}, \mathbf{w}\}$?

Solution. We must solve the equation

$$x\mathbf{v} + y\mathbf{w} = \mathbf{u},$$

so we row reduce:

$$\begin{bmatrix} 1 & 2 & b \\ 2 & 2 & -2b+4 \\ 3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & b \\ 0 & -2 & -4b+4 \\ 0 & -4 & -3b+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & b \\ 0 & -2 & -4b+4 \\ 0 & 0 & 5b-6 \end{bmatrix}.$$

This system is solvable precisely when $5b - 6 = 0$, so

- when $b = \frac{6}{5}$, the vector $\mathbf{u}$ is in Span $\{\mathbf{v}, \mathbf{w}\}$ and
- when $b \neq \frac{6}{5}$, the vector $\mathbf{u}$ is not in Span $\{\mathbf{v}, \mathbf{w}\}$.

□

Extra Credit

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a vector $\mathbf{v}$ such that

- The first coordinate of $\mathbf{v}$ is 7,
- $\mathbf{v}$ is perpendicular to $\mathbf{u}$, and
- $\|\mathbf{v}\| = \sqrt{83}$.

Proof. We must find $y, z \in \mathbb{R}$ such that

$$7 + 2y + z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \sqrt{49 + y^2 + z^2} = \left\| \begin{bmatrix} 7 \\ y \\ z \end{bmatrix} \right\| = \sqrt{83}.$$

These equations imply

$$z = -7 - 2y \quad \text{and} \quad 49 + y^2 + z^2 = 83.$$

Substituting, we obtain

$$49 + y^2 + (-7 - 2y)^2 = 83,$$

which simplifies to

$$(5y + 3)(y + 5) = 5y^2 + 28y + 15 = 0,$$

so the only possibilities are $y = -5$ and $y = -3/5$. Since $z = -7 - 2y$, we obtain two candidates for $\mathbf{v}$:

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ -5 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ -3/5 \\ -29/5 \end{bmatrix}.$$

By direct computation we see that

$$\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{83} \quad \text{and} \quad \mathbf{v}_1 \cdot \mathbf{u} = \mathbf{v}_2 \cdot \mathbf{u} = 0,$$

as desired. □