Math 261 Section 6 Fall 2019 Exam 1 October 23, 2019

Name:____

- Put your name in the "____" above.
- Write your answers down neatly, use complete sentences, and *justify your work*.
- Good luck!

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}.$$

Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find all solutions to the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

Answer. We solve by row reducing:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 5 \\ 7 & 8 & 9 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, all solutions are of the form

$$\mathbf{x} = \begin{bmatrix} t \\ 1 - 2t \\ t \end{bmatrix}$$

□

2. Suppose that

- B is a 2×3 matrix,
- C is a 3×2 matrix,
- D is a 3×3 matrix, and
- E is a 2×1 matrix.

For each of the following matrix expressions, either tell me the size of the matrix or write "Undefined."

(a) BC

Answer. This is a 2×2 matrix. □

(b) CB

Answer. This is a 3×3 matrix. □

(c) $B + C$

Answer. Undefined. □

(d) $C + B$

Answer. Undefined. □

(e) BD

Answer. This is a 2×3 matrix. $\square$

(f) BE

Answer. Undefined. $\square$

(g) BDC

Answer. This is a 2×2 matrix. $\square$

3. Let

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 6 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find F^{-1}, if it exists.

Proof. We solve by row reducing:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -7 & -6 & -4 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -7 & -6 & -4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 & -1 & 9 \\ 0 & 1 & 0 & -2 & -1 & 6 \\ 0 & 0 & 1 & 3 & 1 & -7 \end{bmatrix},$$

so

$$F^{-1} = \begin{bmatrix} -4 & -1 & 9 \\ -2 & -1 & 6 \\ 3 & 1 & -7 \end{bmatrix}.$$

$\square$

4. (a) Write a system of two linear equations in two variables that has infinitely many solutions.

Answer. One example is:

$$\begin{aligned} x + y &= 0 \\ x + y &= 0. \end{aligned}$$

$\square$

(b) Solve your system from part (a).

Answer. We solve by row reducing:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

$\square$

so the solutions to my example from (a) are of the form

$$\begin{aligned} x &= -t \\ y &= t. \end{aligned}$$

5. Suppose that a is a real number, and let

$$G = \begin{bmatrix} a & 2a \\ 3a & 4a+2 \end{bmatrix}.$$

For which values of a is G invertible?

Answer. If $a = 0$, then the first column is all 0s, so there is no pivot in the first column of $[G|I]$, and we see that G is not invertible in this case.

On the other hand, if $a \neq 0$, we solve by row reducing:

$$\begin{bmatrix} a & 2a & 1 & 0 \\ 3a & 4a+2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \frac{1}{a} & 0 \\ 3a & 4a+2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \frac{1}{a} & 0 \\ 0 & -2a+2 & -3 & 1 \end{bmatrix}.$$

This system is solvable precisely when $-2a + 2 \neq 0$—that is, when $a \neq 1$.

Thus,

- when $a \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$, the matrix G is invertible and
- when $a = 0, 1$, the matrix G is not invertible.

□

6. Let

$$H = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) Let E_1 be the elementary matrix associated to scaling row two of H by 2. What is E_1?

Answer.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(b) Let E_2 be the elementary matrix associated to adding $(-1) \cdot$ (row one) to row two. What is E_2?

Answer.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(c) Let E_3 be the elementary matrix associated to adding $(-1) \cdot$ (row three) to row two. What is E_3?

Answer.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(d) What is $E_3E_2E_1H$?

Answer. We perform the row operations mentioned above to obtain

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{bmatrix}.$$

□

Extra Credit

Let

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Write down a 9×9 matrix J such that

$$J\mathbf{x} = \mathbf{b} \text{ has infinitely many solutions} \qquad \text{and} \qquad J\mathbf{x} = \mathbf{b} \text{ has } \mathbf{x}_0 \text{ as a solution.}$$

Answer. One example is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\square$