Math 261 Fall 2024 Exam Two Solutions

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Let

$$A = \begin{bmatrix} 4 & 2a & a \\ 0 & 0 & 1 \\ a & 2 & 3 \end{bmatrix},$$

where a is a real number. Consider the matrix equation

$$A\mathbf{x} = \mathbf{0}.$$

For which values of a does this equation have:

(a) no solutions,

(b) exactly one solution, and

(c) infinitely many solutions?

Solution. Since the matrix is homogeneous, it always has at least one solution. It has infinitely many solutions precisely when A is not invertible. Since $\det(A) = -\left(8 - 2a^2\right) = 2(a+2)(a-2)$, we conclude that the equation

(a) never has no solutions,

(b) has exactly one solution when $a \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$, and

(c) infinitely many solutions when $a \in \{-2, 2\}$.

□

2. Let

$$C = \begin{bmatrix} 1 & -4 \\ 2 & 7 \end{bmatrix}.$$

(a) Find all the eigenvalues of C.

(b) For each eigenvalue of C, find an associated eigenvector of C.

Solution.

Since $\det(C - \lambda I) = (1-\lambda)(7-\lambda) + 8 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$, we know C has exactly two eigenvalues: 3 and 5.

Since

$$C - 3I = \begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix},$$

one eigenvector for $\lambda = 3$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and since

$$C - 5I = \begin{bmatrix} -4 & -4 \\ 2 & 2 \end{bmatrix},$$

one eigenvector for $\lambda = 5$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. □

3. Suppose that $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Define a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = \operatorname{proj}_{\mathbf{v}}(\mathbf{x}).$$

Let B be the matrix for T.

(a) What is B?

(b) What is the rank of B?

Solution. (a) We compute

i. $T(\mathbf{e}_1) = \frac{\mathbf{e}_1 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{0}{2} \mathbf{v} = \mathbf{0}$,

ii. $T(\mathbf{e}_2) = \frac{\mathbf{e}_2 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1}{2} \mathbf{v}$,

iii. $T(\mathbf{e}_3) = \frac{\mathbf{e}_3 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1}{2} \mathbf{v}$,

so

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) The rank of B is one. (In fact, since the range of T is span $\{\mathbf{v}\}$, we can come to this conclusion without completing part (a).)

□

4. (a) Write down a 2×2 matrix with exactly 2 distinct eigenvalues.

(b) Write down a 2×2 matrix with no eigenvalues.

(c) Write down a 3×3 matrix with exactly 2 distinct eigenvalues.

(d) Write down a 4×4 matrix with exactly 2 distinct eigenvalues.

Solution. For example, the following matrices would work:

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{(b)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \text{(c)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \text{and} \quad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

5. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and let

$$U = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{x} = 0 \right\} \quad \text{and} \quad V = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{x} = 0 \textit{ and } \mathbf{v} \cdot \mathbf{x} = 0 \right\}.$$

Find bases for U and V.

Solution.

- For U, we must solve a homogeneous matrix equation, so we row reduce $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. This system as two free variables, so a basis for U is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- For V, we must solve a system of two equations, so we row reduce

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

This system has one free variable, so a basis for V is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

□

6. Let

- R be the plane in $\mathbb{R}^3$ given by the equation $x + 2y + 3z = 0$,
- S be the plane in $\mathbb{R}^3$ given by the plane $x + y + z = 0$,
- L be the line with direction $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ that contains the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, and
- M the line of intersection of the planes R and S.

(a) Where does L intersect S?

(b) What is M?

(c) Do L and M intersect? If so, where do they intersect?

Solution. (a) We must find t such that $\begin{bmatrix} -t+5 \\ t+5 \\ -t+5 \end{bmatrix}$ satisfies $x + y + z = 0$; that is, such that

$$(-t+5) + (t+5) + (-t+5) = 0.$$

We find $t = 15$, so L and S intersect at

$$15 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \\ -10 \end{bmatrix}.$$

(b) In fact, we found the intersection of these planes in Question 5. The line is

$$\operatorname{span}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\} = \left\{t\begin{bmatrix}1\\-2\\1\end{bmatrix} \;\middle|\; t \in \mathbb{R}\right\}.$$

(c) We ask the question: are there $s, t \in \mathbb{R}$ such that

$$s\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \begin{bmatrix}5\\5\\5\end{bmatrix} = t\begin{bmatrix}1\\-2\\1\end{bmatrix}?$$

To answer this question, we row reduce the augmented matrix

$$\begin{bmatrix}-1 & -1 & -5\\1 & 2 & -5\\-1 & -1 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 5\\0 & 1 & -10\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 15\\0 & 1 & -10\\0 & 0 & 0\end{bmatrix}$$

to find $s = 15, t = -10$. Thus, the point of intersection is

$$15\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \begin{bmatrix}5\\5\\5\end{bmatrix} = \begin{bmatrix}-10\\20\\-10\end{bmatrix} = (-10)\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

□

7. Find two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ that are

- on the plane $x + y + z = 0$,
- on the plane $x = 1$, and
- are of length $\sqrt{14}$.

Solution. Let's find $a, b, c \in \mathbb{R}$ such that $\begin{bmatrix}a\\b\\c\end{bmatrix}$ satisfy the three conditions given in the problem. The second bullet tells us that $a = 1$, and the first tells us that $c = -b - 1$. Thus, we must find b such that

$$\sqrt{14} = \sqrt{1^2 + b^2 + (-b-1)^2}.$$

Squaring, we solve

$$14 = 1^2 + b^2 + (-b-1)^2 = 2b^2 + 2b - 12 = 2(b+3)(b-2),$$

and find $b = -3, 2$. Thus, two vectors that satisfy the requirements of the problem are

$$\begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}1\\2\\-3\end{bmatrix}.$$

□