Math 261 Fall 2024 Exam Two

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Let

$$A = \begin{bmatrix} 4 & 2a & a \\ 0 & 0 & 1 \\ a & 2 & 3 \end{bmatrix},$$

where a is a real number. Consider the matrix equation

$$A\mathbf{x} = \mathbf{0}.$$

For which values of a does this equation have:

(a) no solutions,

(b) exactly one solution, and

(c) infinitely many solutions?

2. Let

$$C = \begin{bmatrix} 1 & -4 \\ 2 & 7 \end{bmatrix}.$$

(a) Find all the eigenvalues of C.

(b) For each eigenvalue of C, find an associated eigenvector of C.

3. Suppose that $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = \operatorname{proj}_{\mathbf{v}}(\mathbf{x}).$$

Let B be the matrix for T.

(a) What is B?

(b) What is the rank of B?

4. (a) Write down a 2×2 matrix with exactly 2 distinct eigenvalues.
   (b) Write down a 2×2 matrix with no eigenvalues.
   (c) Write down a 3×3 matrix with exactly 2 distinct eigenvalues.
   (d) Write down a 4×4 matrix with exactly 2 distinct eigenvalues.

5. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and let

$$U = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{x} = 0 \right\} \quad \text{and} \quad V = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{x} = 0 \textit{ and } \mathbf{v} \cdot \mathbf{x} = 0 \right\}.$$

Find bases for U and V.

6. Let

- R be the plane in $\mathbb{R}^3$ given by the equation $x + 2y + 3z = 0$,
- S be the plane in $\mathbb{R}^3$ given by the plane $x + y + z = 0$,
- L be the line with direction $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ that contains the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, and
- M the line of intersection of the planes R and S.

(a) Where does L intersect S?

(b) What is M?

(c) Do L and M intersect? If so, where do they intersect?

7. Find two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ that are

- on the plane $x + y + z = 0$,
- on the plane $x = 1$, and
- are of length $\sqrt{14}$.