Math 261 Fall 2024 Exam One Solutions

Name:____

- Put your name in the "____" above.
- Answer all questions.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Solve the following system of linear equations.

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + 3z &= -2 \\ 4x + 3y + 5z &= -5. \end{aligned}$$

Solution. We row reduce the associated augmented matrix as follows:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 3 & -2 \\ 4 & 3 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & -3 \\ 0 & 1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & 3 & -7 \\ 2 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & 3 & -7 \\ 0 & -1 & -3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so

$$\begin{aligned} x &= z + 4 \\ y &= -3z - 7 \\ z &= z. \end{aligned}$$

In terms of a parameter t, the solutions to the system of linear equations have the form

$$\begin{aligned} x &= t + 4 \\ y &= -3t - 7 \\ z &= t \end{aligned}$$

for any $t \in \mathbb{R}$. □

2. Define a function $T: \mathbb{R}^3 \to \mathbb{R}^3$ by the rule

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ z \\ y \end{bmatrix}.$$

(a) Compute

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

(b) You may assume that T is a linear transformation. Find a matrix A such that for all $\mathbf{x} \in \mathbb{R}^3$,

$$T(\mathbf{x}) = A\mathbf{x}.$$

(c) Find all vectors $\mathbf{u}$ with the property that

$$T(\mathbf{u}) = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

Solution. (a) $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) Solving $A\mathbf{x} = \mathbf{b}$ gives the unique solution $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

□

3. (a) Write a system of three linear equations in three variables that has infinitely many solutions.
   (b) Solve your system from part (a).

Solution. (a) Consider

$$\begin{aligned} x &= 0 \\ x &= 0 \\ x &= 0. \end{aligned}$$

(b) The solutions are

$$\begin{aligned} x &= 0 \\ y &= s \\ z &= t. \end{aligned}$$

for any choice of s, t.

□

4. Suppose that A, B, C are matrices.

   (a) Suppose that A is 3×8, that C is 3×15, and that $AB = C$. What is the size of B?
   (b) Suppose that A is 10×10. What is the size of A^{99}?
   (c) Suppose that A is 1×2, that B is 2×3 and C is 3×4. What is the size of ABC?

Solution. (a) 8×15

(b) 10×10

(c) 1×4

□

5. For which real numbers a is the following matrix invertible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & a & 2 \\ 0 & 3 & a \end{bmatrix}$$

Solution. We compute a determinant:

$$\det \begin{bmatrix} 2 & 1 & 0 \\ 2 & a & 2 \\ 0 & 3 & a \end{bmatrix} = \det \begin{bmatrix} 2 & 1 & 0 \\ 0 & a-1 & 2 \\ 0 & 3 & a \end{bmatrix} = 2((a-1)a-6) = 2(a-3)(a+2),$$

so A is invertible when $a \in (-\infty, -2) \cup (-2, 3) \cup (3\infty)$. □

6. Write the following matrix as a product of elementary matrices. (You don't need to write all the 0s in your matrices.)

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. Since B is so close to an elementary matrix itself, we can eyeball this:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

□

7. Let a be a real number,

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}.$$

For which values of a is $\mathbf{w}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

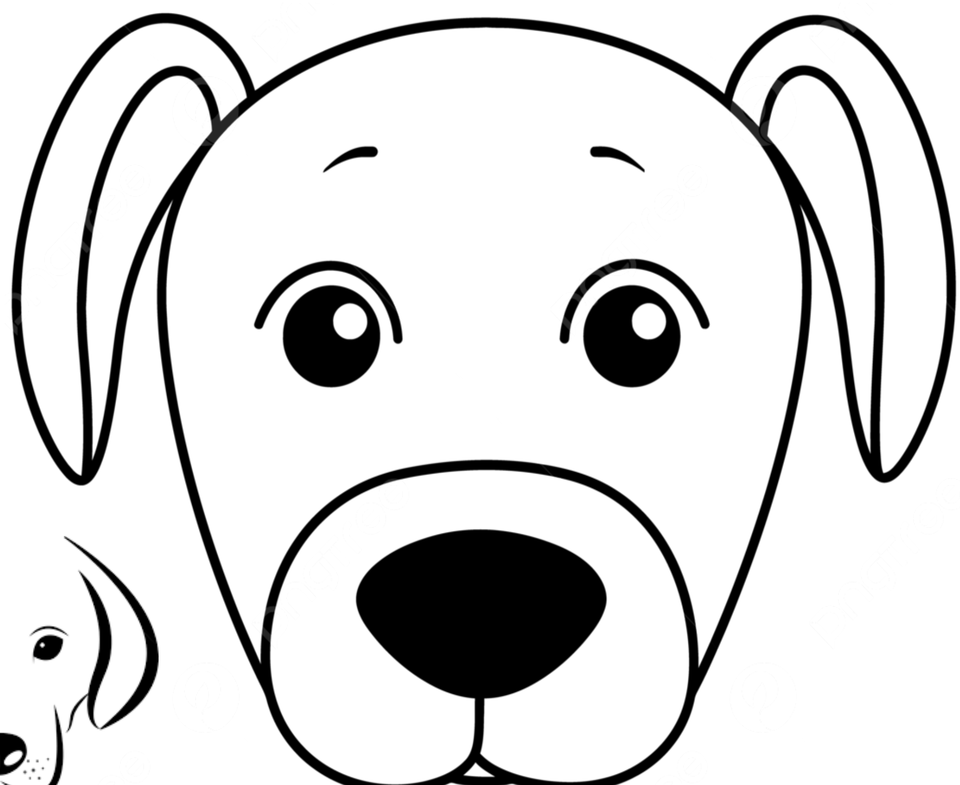
Solution. We can see that $\mathbf{w}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$ if and only if $a = 0$. □

Extra credit

Suppose that Adira, Badele, and Connor are three cute dogs. We know that

- Badele weighs twice as much as Adira,
- Connor's weight is the sum of Adira's weight and Badele's weight, and
- Connor weighs 80 pounds more than Adira.

Name the dogs based on their size.



name:Adira name:Badele name:Connor