Math 261 Section 6 Fall 2019 Final Exam December 12, 2019

Name:____

- Put your name in the "____" above.
- Write your answers down neatly, use complete sentences, and *justify your work*.
- The extra credit question is for your amusement, if you have extra time at the end of the test.
- Good luck!

1. Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that for all $x, y, z \in \mathbb{R}$,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2x+4z \end{bmatrix}.$$

(a) Find the matrix of T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

(c) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$.

Solution. (a) Since

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

we see that the matrix for T is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}.$$

(b) $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$.

(c) We row reduce the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 0 & 4 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \end{bmatrix},$$

so

$$\left\{\mathbf{x} \in \mathbb{R}^3 \mid T(\mathbf{x}) = \begin{bmatrix} 6 \\ 14 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R}\right\}.$$

□

2. Let

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Is $\mathbf{u}$ an eigenvector for A? If it is, what is its corresponding eigenvalue? (Hint: you do not need to compute the characteristic polynomial for A.)

(b) Is $\mathbf{v}$ an eigenvector for A? If it is, what is its corresponding eigenvalue? (Hint: you do not need to compute the characteristic polynomial for A.)

Solution. (a) Since

$$A\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

is not a multiple of $\mathbf{u}$, we see that $\mathbf{u}$ is not an eigenvector for A.

(b) Since

$$A\mathbf{v} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\mathbf{v},$$

we see that $\mathbf{v}$ is an eigenvector for A with eigenvalue 5.

□

3. Let

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of B.

(b) For each eigenvalue of B, find a basis for its associated eigenspace.

Proof. (a) Since solutions to

$$0 = \det(B - \lambda I) = (2 - \lambda)^2(3 - \lambda)$$

are $\lambda = 2, 3$, we see that B has eigenvalues: 2 and 3.

(b) First, we solve the augmented matrix associated to $(B - 2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which has solutions

$$\{t\mathbf{e}_1 \mid t \in \mathbb{R}\}.$$

And next, we solve the augmented matrix associated to $(B - 3I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which has solutions

$$\{t\mathbf{e}_3 \mid t \in \mathbb{R}\}.$$

□

4. (a) Write down a set of four distinct vectors in $\mathbb{R}^3$ that do not span $\mathbb{R}^3$.
   (b) Write down a 3×5 matrix of rank 2.
   (c) Write down a 3×5 matrix of nullity 3.
   (d) Write down a 2×2 matrix with an eigenvalue of 3.

Solution.

$$\text{(a)}\ \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}3\\0\\0\end{bmatrix}\right\} \qquad \text{(b) and (c)}\ \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix} \qquad \text{(d)}\ \begin{bmatrix}3&0\\0&3\end{bmatrix}$$

□

5. Let

$$C = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}.$$

   (a) Find a basis for the column space of C, then write down the rank of C.
   (b) Find a basis for the null space of C, then write down the nullity of C.

Solution. First we row reduce C:

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \sim \begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix} \sim \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}.$$

So then:

(a) C has rank 2, and a basis for the column space of C is

$$\left\{\begin{bmatrix}1\\4\\7\end{bmatrix},\begin{bmatrix}2\\5\\8\end{bmatrix}\right\}.$$

(b) C has nullity 1, and a basis for the null space of C is

$$\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}.$$

□

6. Let

$$V = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a+b=c \text{ and } a+c=d\right\}.$$

   (a) Show that V is a subspace of $\mathbb{R}^4$ by writing it as a span.
   (b) Find a basis for V, then write down the dimension of V.

Solution. (a) Note that

$$V = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a+b=c \text{ and } a+c=d\right\} = \left\{\begin{bmatrix}a\\b\\a+b\\2a+b\end{bmatrix} \in \mathbb{R}^4\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right\},$$

so that V is a subspace of $\mathbb{R}^3$.

(b) By the previous part, we see that $\dim V = 2$ and a basis for V is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

□

7. Suppose that a is a real number. Find all value(s) of a for which the following matrix is invertible:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3a \\ 4 & a+7 & a+9 \end{bmatrix}.$$

Solution. We compute

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3a \\ 4 & a+7 & a+9 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 3a-9 \\ 0 & a-1 & a-3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -3 & 3a-9 \\ a-1 & a-3 \end{bmatrix}\right) = -3(a-3)-(a-1)(3a-9) = -3a(a+3),$$

so the matrix is invertible when $a \in (-\infty, -3) \cup (-3, 0) \cup (0, \infty)$. □

8. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Write down a nonzero vector in $\mathbb{R}^4$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(b) Find all vectors $\mathbf{x} \in \mathbb{R}^4$ such that $\mathbf{x}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. (Hint: the condition that $\mathbf{x}$ is perpendicular to another vector is equivalent to the condition that the coefficients of $\mathbf{x}$ satisfy a certain linear equation—so to solve this question, you must solve a system of linear equations.)

Solution. (a) One example is $\begin{bmatrix} 0 \\ 0 \\ 4 \\ -3 \end{bmatrix}$.

(b) We solve the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -3 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & -1 & -3 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 3 & 4 & 0 \end{bmatrix},$$

which has solutions

$$\left\{ s\begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}, t\begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

□