Math 261 Section 6 Fall 2019 Final Exam December 12, 2019

Name:____

- Put your name in the "____" above.
- Write your answers down neatly, use complete sentences, and *justify your work*.
- The extra credit question is for your amusement, if you have extra time at the end of the test.
- Good luck!

1. Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that for all $x, y, z \in \mathbb{R}$,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2x + 4z \end{bmatrix}.$$

(a) Find the matrix of T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

(c) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$.

2. Let

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Is **u** an eigenvector for A? If it is, what is its corresponding eigenvalue? (Hint: you do not need to compute the characteristic polynomial for A.)

(b) Is **v** an eigenvector for A? If it is, what is its corresponding eigenvalue? (Hint: you do not need to compute the characteristic polynomial for A.)

3. Let

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of B.

(b) For each eigenvalue of B, find a basis for its associated eigenspace.

4. (a) Write down a set of four distinct vectors in $\mathbb{R}^3$ that do not span $\mathbb{R}^3$.

(b) Write down a 3×5 matrix of rank 2.

(c) Write down a 3×5 matrix of nullity 3.

(d) Write down a 2×2 matrix with an eigenvalue of 3.

5. Let

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(a) Find a basis for the column space of C, then write down the rank of C.

(b) Find a basis for the null space of C, then write down the nullity of C.

6. Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a + b = c \text{ and } a + c = d \right\}.$$

(a) Show that V is a subspace of $\mathbb{R}^4$ by writing it as a span.

(b) Find a basis for V, then write down the dimension of V.

7. Suppose that a is a real number. Find all value(s) of a for which the following matrix is invertible:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3a \\ 4 & a+7 & a+9 \end{bmatrix}.$$

8. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Write down a nonzero vector in $\mathbb{R}^4$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(b) Find all vectors $\mathbf{x} \in \mathbb{R}^4$ such that $\mathbf{x}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. (Hint: the condition that $\mathbf{x}$ is perpendicular to another vector is equivalent to the condition that the coefficients of $\mathbf{x}$ satisfy a certain linear equation—so to solve this question, you must solve a system of linear equations.)