Using puzzles to solve problems

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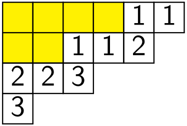
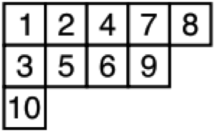
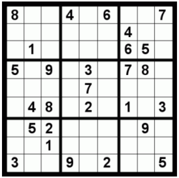
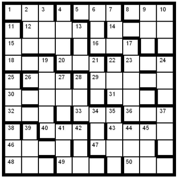




Have you ever played a “math puzzle”





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Today, we are going to explore these puzzles and uncover some difficult problems they solve!

The space of symmetric functions with n variables

- $f(x_1, ..., x_n)$ is a *symmetric function* with n variables if
  $f(x_1, ..., x_n) = f(x_{i_1}, ..., x_{i_n})$ where $x_{i_1}, ..., x_{i_n}$ is any rearrangement of the variables $x_1, ..., x_n$.

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Example (three variables)

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Example (three variables)

$$f(x_1, x_2, x_3) = x_1 x_2 x_3,$$

$$g(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

Question

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What is a basis for the algebra of symmetric functions with n variables?

Yes! They are called Schur polynomials!

Definition of Schur Polynomial

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Definition

For each partition λ, the Schur polynomial is defined as

$$s_\lambda(\bar{x}) = \frac{\left| x_i^{n+\lambda_j - j} \right|_{1 \leq i,j \leq n}}{\left| x_i^{n-j} \right|_{1 \leq i,j \leq n}}$$

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$$s_\lambda(x_1, x_2) = \frac{\begin{vmatrix} x_1^{2+\lambda_1-1} & x_1^{2+\lambda_2-2} \\ x_2^{2+\lambda_1-1} & x_2^{2+\lambda_2-2} \end{vmatrix}}{\begin{vmatrix} x_1^{2-1} & x_1^{2-2} \\ x_2^{2-1} & x_2^{2-2} \end{vmatrix}}$$

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$$s_\lambda(x_1, x_2) = \frac{\begin{vmatrix} x_1^4 & x_1^1 \\ x_2^4 & x_2^1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_1^0 \\ x_2 & x_2^0 \end{vmatrix}}$$

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$$s_\lambda(x_1, x_2) = \frac{\begin{vmatrix} x_1^4 & x_1 \\ x_2^4 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \frac{x_1^4 \cdot x_2 - x_1 \cdot x_2^4}{x_1 - x_2}$$

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$$s_\lambda(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3$$

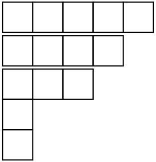
Schur Polynomial- Using Young tableaux

There is another way to compute $s_\lambda(\bar{x})$ using puzzles!

Young Diagram (shape)

For a partition $\lambda = (\lambda_1 \geq, ... \geq \lambda_n \geq 0)$ we can create a Young diagram:

$\lambda =$



$\leftarrow \lambda_1$
$\leftarrow \lambda_2$
$\leftarrow \dots$
$\leftarrow \dots$
$\leftarrow \lambda_n$

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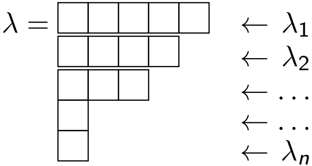

The total number of boxes, $|\lambda| = \sum_{i=1}^{n} \lambda_i$, is called the *weight* of λ.

Examples: Young diagram

Example

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$\lambda = (3, 1)$

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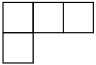
$\lambda =$ 

Examples: Young diagram

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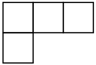
$\lambda =$ 

Example

$\mu = (4, 3, 1, 1)$

Examples: Young diagram

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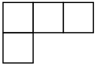
$\lambda =$ 

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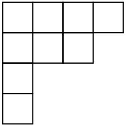
$\mu =$ 

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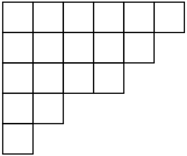
  A *Young tableau* with shape $\lambda = (\lambda_1, ..., \lambda_n)$ and content $\mu = (\mu_1, ..., \mu_n)$ is a filling of

$\lambda =$



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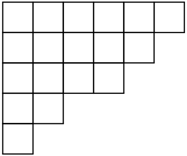
$\lambda =$ 

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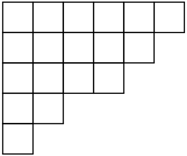
$\lambda =$ 

with μ_1 1's, μ_2 2's, ..., μ_n n's (referred to as *flavors*) such that flavors

- weakly increase across rows and
- strictly increase down columns.

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- $\lambda = (4, 3, 1, 1)$ and $\mu = (2, 3, 3, 1)$,

Example: Young tableau

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- $\lambda = (4, 3, 1, 1)$ and $\mu = (2, 3, 3, 1)$,
- $|\lambda| = 9$, $|\mu| = 9$.

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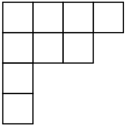
- Fill $\lambda =$  with flavors $1, 1, 2, 2, 2, 3, 3, 3, 4$ so that,

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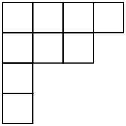
- Fill $\lambda =$  with flavors $1, 1, 2, 2, 2, 3, 3, 3, 4$ so that,
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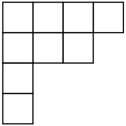
- Fill $\lambda =$  with flavors $1, 1, 2, 2, 2, 3, 3, 3, 4$ so that,
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Example

1	1	2	2
2	3	3	
3			
4			

Example: Young tableau

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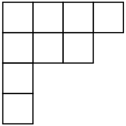
- Fill $\lambda =$  with flavors $1, 1, 2, 2, 2, 3, 3, 3, 4$ so that,
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Non Example!

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2			
4			

Example: Young tableau

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- $\lambda = (4, 3, 1, 1)$ and $\mu = (2, 3, 3, 1)$
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Definition of Schur Polynomial

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Schur polynomials- Using Young tableaux

There is another way to compute using Young tableaux.

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where

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- $\bar{x}^\mu$ is the monomial $x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$.

Example: Schur Polynomial

Let's compute $s_\lambda(x_1, x_2)$, with $\lambda = (3, 1)$ using Young tableaux!

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$$s_\lambda(\bar{x}) = \sum K_{\lambda,\mu}\bar{x}^\mu$$

where

- $\mu = (\mu_1, \mu_2)$ is a collection of 2 integers with $|\mu| = \mu_1 + \mu_2 = |\lambda| = 4$.

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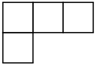
- $K_{\lambda,\mu}$ is the number of Young tableaux with shape $\lambda =$  and content μ.

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a) What are all possible $\mu = (\mu_1, \mu_2)$ such that $\mu_1 + \mu_2 = 4$?

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$$s_\lambda(x_1, x_2) = \sum K_{\lambda,\mu} \bar{x}^\mu$$

a) What are all possible $\mu = (\mu_1, \mu_2)$ such that $\mu_1 + \mu_2 = 4$?

$\mu = (4, 0), (3, 1), (2, 2), (1, 3), (0, 4).$

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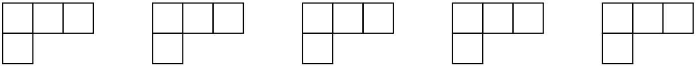

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1	1	1
2		

1	1	2
2		

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2		

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Compare with determinant definition?

Computing

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$$s_\lambda(x_1, x_2) = \frac{\begin{vmatrix} x_1^4 & x_1 \\ x_2^4 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_1 \\ x_2 & x_2 \end{vmatrix}} = \frac{x_1^4 \cdot x_2 - x_1 \cdot x_2^4}{x_1 - x_2} =$$

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$$= \frac{(x_1 - x_2)(x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3)}{x_1 - x_2} = x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3$$

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Multiplying Schur Polynomials

Question

Is $s_\lambda(\bar{x}) \cdot s_\mu(\bar{x})$ *still a symmetric function?*

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Question

Is $s_\lambda(\bar{x}) \cdot s_\mu(\bar{x})$ still a symmetric function?

Yes! It is a sum of Schur polynomials with non-negative coefficients.

Multiplying Schur Polynomials

We can write the product as:

$$s_\mu(\bar{x})s_\nu(\bar{x}) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(\bar{x}). \tag{2}$$

where the summation is over all partitions $\lambda = (\lambda_1, ..., \lambda_n)$ with $|\lambda| = |\mu| + |\nu|$.

LR-Coefficient

The coefficient $c_{\mu\nu}^{\lambda}$ is called a *Littlewood-Richardson coefficient* (1934).

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How do we compute $c_{\mu\nu}^{\lambda}$??

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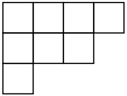
- when reading the flavors in Young tableau from right to left across rows and top to bottom down columns, at any stage, $\#1's \geq \#2's \geq \cdots \geq \#n's$
- we will call such a filling a *reverse lattice word*.

Example of computing LR-coefficient

For $\lambda = (4,3,1)$, $\mu = (3,2,0)$, $\nu = (2,1,0)$, let's compute $c^{\lambda}_{\mu,\nu}$.

Young diagram shape λ/μ

$\lambda =$ 

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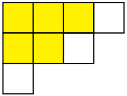
$\lambda/\mu =$ 

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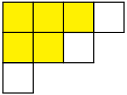
Fill this shape with $1,1,2$ so that filling is a *reverse lattice word* (at any stage while reading off word, we have $\#1 \geq \#2$).

$\lambda/\mu =$ 

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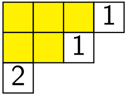

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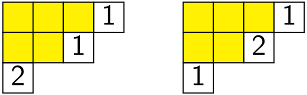

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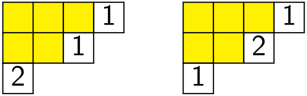

$c^{(4,3,1)}_{(3,2,0),(2,1,0)} = 2.$

Challenge: Computing LR-coefficient

For $\lambda = (6, 5, 3, 1)$, $\mu = (4, 2, 0, 0)$, $\nu = (4, 3, 2, 0)$, let's compute $c^{\lambda}_{\mu,\nu}$.

$c^{\lambda}_{\mu\nu}$ is the number of Young tableaux with shape λ/μ and content ν with the condition that

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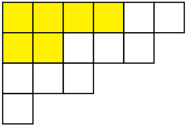
$\lambda/\mu =$ 

Fill with $1, 1, 1, 1, 2, 2, 2, 3, 3$ so that the filling is a reverse lattice word. How many are there?

Challenge Solutions

Challenge Solutions

				1	1
		1	1	2	
2	2	3			
3					

Challenge Solutions

				1	1
		1	1	2	
2	2	3			
3					

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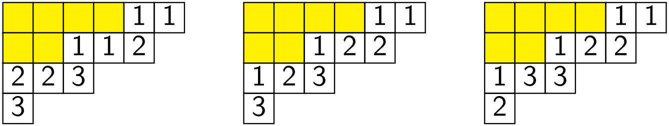

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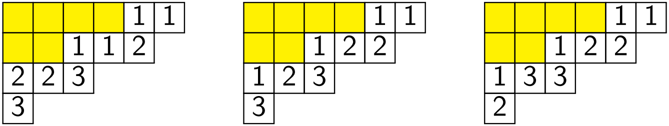
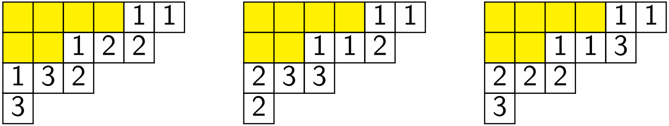


$c^{\lambda}_{\mu\nu} = 3$

Challenge: Non-Solutions



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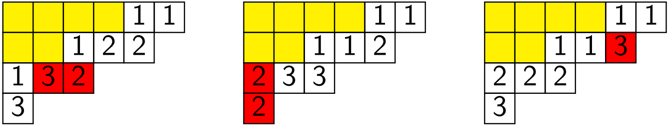

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Question

When is $dim(\mathbb{V}(\mathfrak{sl}_2, \lambda, \ell)) = 1$?

Littlewood-Richardson and Vector Bundles of Conformal Blocks [1]

Let k and p be integers such that $|\lambda| = \sum_{i=1}^{n} \lambda_i = 2(k\ell + p)$, then

$$\mathrm{rank}(\mathbb{V}(\mathfrak{sl}_2, \lambda, \ell)) = \#\Big\{ \text{YT with shape } \rho = (\ell^{2k}, p^2) \text{ and content } (\lambda_1, ..., \lambda_n)$$

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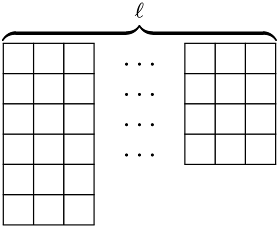
$\rho =$ 

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Example: Computing ranks

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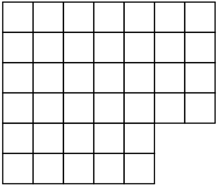
rank($\mathbb{V}$) = $\#\{$YT on  and content $(7,7,6,5,5,4,4)\}$.

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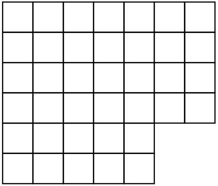

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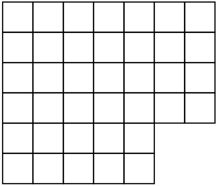

and content (7, 7, 6, 5, 5, 4, 4).

Fill the above boxes with the flavors
$1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4,$
$4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7.$

Example of computing rank

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	5
5	5	6	6	6		
6	7	7	7	7		

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2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	5
5	5	6	6	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	6
5	5	5	6	6		
6	7	7	7	7		

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1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	5
5	5	6	6	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	6	6
5	5	5	5	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	6
5	5	5	6	6		
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2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	5
5	5	6	6	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	6	6
5	5	5	5	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	6
5	5	5	6	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	5
4	4	4	4	4	6	6
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2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	5	5
5	5	6	6	6		
6	7	7	7	7		

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
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5	5	5	6	6		
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1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	4	4	5	6	6
5	5	5	5	6		
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1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	5
4	4	4	4	4	6	6
5	5	5	5	6		
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3	3	3	3	3	3	5
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Classification of $\mathbb{V}(\mathfrak{sl}_2)$ ranks

Theorem

Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (\lambda_1, ..., \lambda_n), \ell)$ *with* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$*,*
$\sum_{i=1}^{n} \lambda_i = 2(k\ell + p)$*, and for some* p*,* k *such that* $0 \leq p < \ell$ *and* $k \geq 0$*.*
Denote $\Lambda = \sum_{i=2k+2}^{n} c_i$*. Then*

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- $dim(\mathbb{V}) = 0$ *iff* $\Lambda < p$;

Classification of $\mathbb{V}(\mathfrak{sl}_2)$ ranks

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Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (\lambda_1, ..., \lambda_n), \ell)$ *with* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$*,* $\sum_{i=1}^{n} \lambda_i = 2(k\ell + p)$*, and for some* p*,* k *such that* $0 \leq p < \ell$ *and* $k \geq 0$*. Denote* $\Lambda = \sum_{i=2k+2}^{n} c_i$*. Then*

- $dim(\mathbb{V}) = 0$ *iff* $\Lambda < p$*;*
- $dim(\mathbb{V}) = 1$ *iff* $\Lambda = p$*;*

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Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, (\lambda_1, ..., \lambda_n), \ell)$ *with* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, $\sum_{i=1}^{n} \lambda_i = 2(k\ell + p)$, *and for some* p, k *such that* $0 \leq p < \ell$ *and* $k \geq 0$. *Denote* $\Lambda = \sum_{i=2k+2}^{n} c_i$. *Then*

- $dim(\mathbb{V}) = 0$ *iff* $\Lambda < p$;
- $dim(\mathbb{V}) = 1$ *iff* $\Lambda = p$;
- $dim(\mathbb{V}) > 1$ *iff* $\Lambda > p$.

For more details:

- *Quantum Kostka and the rank one problem for* $\mathfrak{sl}_{2m}$
  Hobson, N., arXiv preprint arXiv:1508.06952, 2015.
- *Puzzles Littlewood-Richardson coefficients and Horn inequalities*, Azenhas, O., CMUC, University of Coimbra, 2009.
- *The hive model and the polynomial nature of stretched Littlewood-Richardson coefficients*, King, R., J. Combin. Theory, 2009.

Thank you!

Thank you!

Want to learn more?

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