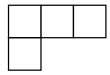
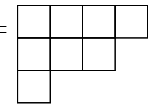
Solving problems with puzzles

Portland State University Math Club
November 25th, 2015

Young Diagram: $\lambda = (3,1)$: $\lambda =$  $\mu = (4,3,1)$: $\mu =$ 

Young Tableau:

Begin with:

- $\lambda = (\lambda_1 \geq ... \geq \lambda_n \geq 0)$ a Young diagram,
- $\mu = (\mu_1, ..., \mu_n)$ a collection of n nonnegative integers such that,
- $|\lambda| = |\mu|$.

A *Young tableau* with shape $\lambda = (\lambda_1, ..., \lambda_n)$ and content $\mu = (\mu_1, ..., \mu_n)$ is a filling of λ with μ_1 1's, μ_2 2's, ..., μ_n n's (referred to as *flavors*) such that flavors
– weakly increase across rows (left to right) and
– strictly increase down columns (top to bottom).

Example:

1	1	2	3
2	2	3	
3			
4			

Non-Example:

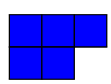
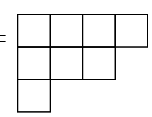
1	1	2	3
2	3	3	
2			
4			

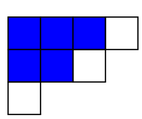
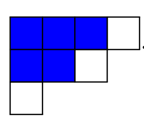
Littlewood-Richardson Coefficients:

The *Littlewood-Richardson Coefficient*, $c^{\lambda}_{\mu\nu}$, is the number of Young tableaux with shape λ/μ and content ν with the condition that

- when reading the flavors in Young tableau from right to left across rows and top to bottom down columns, at any stage, $\#1's \geq \#2's \geq \cdots \geq \#n's$
- we will call such a filling a *reverse lattice word.*

Example:
For $\lambda = (4,3,1)$, $\mu = (3,2,0)$, $\nu = (2,1,0)$, let's compute $c^{\lambda}_{\mu,\nu}$.

First delete the shape $\mu = (3,2,0) =$  from the shape $\lambda = (4,3,1) =$  to get the shape λ/μ Fill this shape with the numbers $1,1,2$ so that filling is a *reverse lattice word* (at any stage while listing this word from right to left and top to bottom we have $\#1 \geq \#2$). Fill in the following two shapes with the slide examples.

$\lambda/\mu =$  .

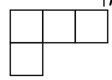
Activities

Exercise 1. Let $\lambda = (3, 1)$ and $\bar{x} = (x_1, x_2)$.
Compute the Schur polynomial $s_\lambda(x_1, x_2)$ using Young tableaux.

To do this, compute the sum:

$$s_{(3,1)}(x_1, x_2) = \sum K_{(3,1),\mu} \bar{x}^\mu$$

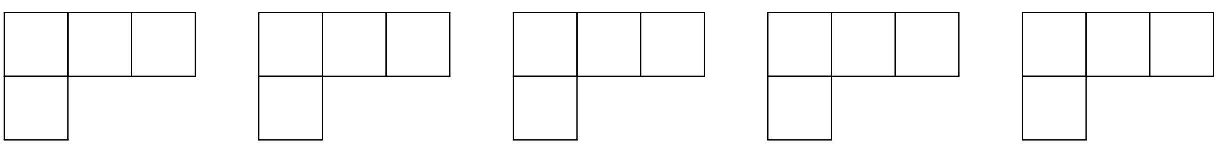
where

- $\mu = (\mu_1, \mu_2)$ is any collection of two nonnegative integers with $|\mu| = \mu_1 + \mu_2 = 4$.
- $K_{(3,1),\mu}$ is the number of Young tableau with shape $\lambda =$  and content μ.
- $\bar{x}^\mu$ is the monomial equal to $\bar{x}^{(\mu_1,\mu_2)} = x_1^{\mu_1} x_2^{\mu_2}$

To compute this sum, follow the steps below.

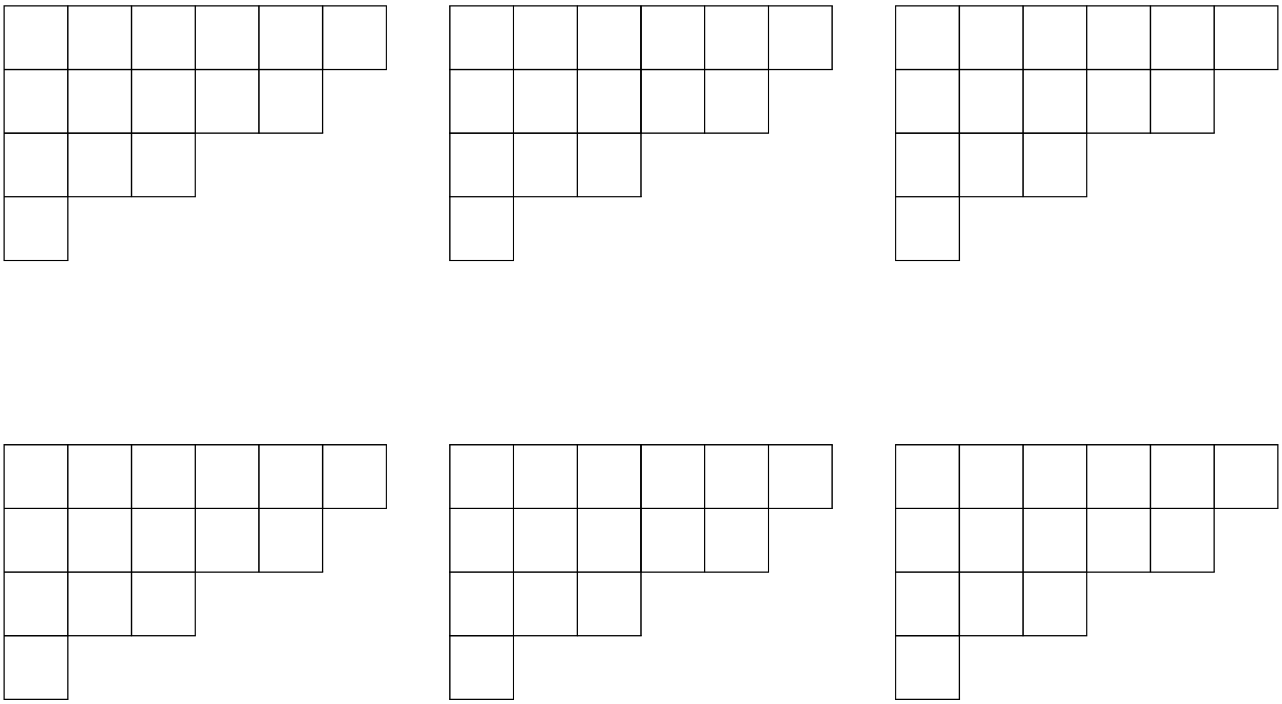
a. What are all possible pairs of nonnegative integers $\mu = (\mu_1, \mu_2)$ such that $|\mu| = \mu_1 + \mu_2 = 4$?

b. For each $\mu = (\mu_1, \mu_2)$ from part (a), how many Young tableaux are there with shape $\lambda = (3, 1)$ and content μ? Use the following outlines to help you construct the Young tableaux. In doing this, you are computing $K_{(3,1),\mu}$ (number of Young tableau with shape $\lambda = (3, 1)$ and content μ).



Exercise 2. Compute $c^{\lambda}_{\mu,\nu}$ with $\lambda = (6,5,3,1)$, $\mu = (4,2,0,0)$, $\nu = (4,3,2,0)$. Use the given shape λ to help you.

Recall: $c^{\lambda}_{\mu\nu}$ is the number of Young tableaux with shape λ/μ and content ν with the condition that when reading the flavors in Young tableau from right to left across rows and top to bottom down columns, at any stage, $\#1's \geq \#2's \geq \#3's \geq \#4's$



How many can you find?

$c^{\lambda}_{\mu,\nu} =$

REFERENCES

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[3] A. Horn, Pacific J. Math **12** (1962), 225-241. ←

[4] A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055-1090. ←

[5] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), 19-48. ←

[6] R. King, *The hive model and the polynomial nature of stretched Littlewood-Richardson coefficients*, J. Combin. Theory, Series A **116** (2009), 314, 333. ←