q-analogs of factorials and Fubini numbers

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- This process is sometimes called "categorification."

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A *q-analog* of $N \in \mathbb{N}$ is a polynomial $f(q) \in \mathbb{N}[q]$ such that

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- We'll see many of examples of what (2) can mean.

An example q-analog: $[n]_q!$

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$$[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q$$

where

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- Clearly $q \to 1$ recovers $n!$, so (1) is satisfied.

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Permutation statistics

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The *symmetric group* of n symbols is

$$\mathfrak{S}_n = \{\text{bijections } \sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\}$$

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- $|\mathfrak{S}_n| = n!$
- Often written in one-line notation.
  - E.g. $\sigma = 52413$ means $\sigma(1) = 5$, $\sigma(2) = 2$, $\sigma(3) = 4$, $\ldots$.

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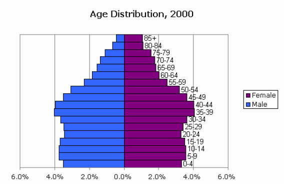

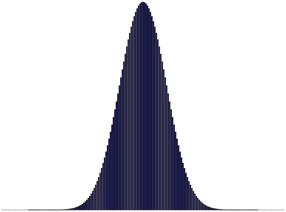

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- For example, $\mathrm{inv}(526413) = 9$.
- $\mathrm{inv}(\sigma)$ also gives the number of adjacent transpositions required to sort σ to the identity permutation.
- How many $\sigma \in \mathfrak{S}_n$ have $\mathrm{inv}(\sigma) = k$ for fixed k?

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Theorem

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and $[3]_q! = 1 + 2q + 2q^2 + q^3$.

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$$8 \mapsto \; 5\ 2\ 3\ 1\ 4\ 7\ 6$$

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Definition [Mac15]

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- Major index depends only on the *descent set* of σ.

MacMahon's Theorem

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- Many other permutation statistics share this distribution.

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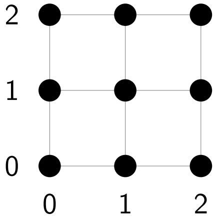

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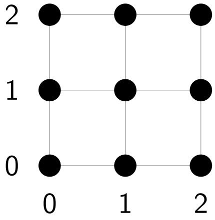

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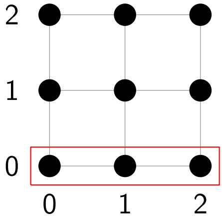

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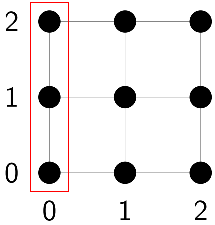

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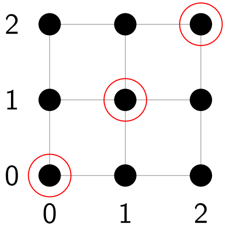

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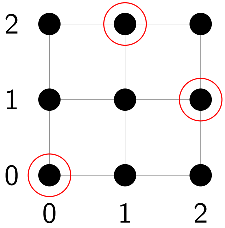

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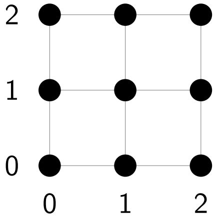
  - $\binom{4}{2} = 6$ bases (from linearly independent lines through the origin)



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- Divide by $n!$ to “unorder” the bases.

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The *Vandermonde matrix* is the matrix

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- δ_n can also be written as

$$\delta_n = \prod_{1 \le i < j \le n} (x_j - x_i)\,.$$

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$$\partial_{x_3}(\delta_3) = (x_2 - x_1)(x_3 - x_2) + (x_2 - x_1)(x_3 - x_1).$$

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- This "decomposes" $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ into a symmetric (invariant) piece and a coinvariant piece.

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Dimension and graded dimension

Theorem [Art42]

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$$\sum_{d \geq 0} \dim\left(\mathbf{H}_n^{(d)}\right) q^d = [n]_q!$$

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- An example basis for $\mathbf{H}_3$:

$$\delta_3 \in \mathbf{H}_3^{(3)}$$

$$\partial_{x_1}(\delta_3), \partial_{x_2}(\delta_3) \in \mathbf{H}_3^{(2)}$$

$$\partial_{x_1}^2(\delta_3), \partial_{x_1}\partial_{x_2}(\delta_3) \in \mathbf{H}_3^{(1)}$$

$$\partial_{x_1}^2\partial_{x_2}(\delta_3) \in \mathbf{H}_3^{(0)}$$

Bases for $\mathbf{H}_n$

- An example basis for $\mathbf{H}_3$:

$$\begin{aligned} \delta_3 &\in \mathbf{H}_3^{(3)} \\ \partial_{x_1}(\delta_3), \partial_{x_2}(\delta_3) &\in \mathbf{H}_3^{(2)} \\ \partial_{x_1}^2(\delta_3), \partial_{x_1}\partial_{x_2}(\delta_3) &\in \mathbf{H}_3^{(1)} \\ \partial_{x_1}^2\partial_{x_2}(\delta_3) &\in \mathbf{H}_3^{(0)} \end{aligned}$$

- Bases can be derived from permutation statistics.

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- What's happening currently?
- One branch is to generalize permutations to *ordered set partitions*.

Ordered set partitions

Definition

An ordered set partition $\pi \in \mathcal{OP}_{n,k}$ is a k-tuple of sets

$$\pi = (\pi_1, \pi_2, \ldots, \pi_k) = \pi_1 | \pi_2 | \ldots | \pi_k$$

such that

$$\bigsqcup_{i=1}^{k} \pi_i = \{1, 2, \ldots, n\}.$$

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- For example, $245|3|16 \in \mathcal{OP}_{6,3}$.
- Note that $\mathcal{OP}_{n,n} = \mathfrak{S}_n$.
- $\mathcal{OP}_{n,k}$ corresponds to surjections

$$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}.$$

Counting ordered set partitions

- $|\mathcal{OP}_{n,k}| = k!S_{n,k}$, where $S_{n,k}$ is the Stirling number of the second kind, defined recursively by

$$S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}.$$

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$$\begin{array}{ccccc}
1 & & & & \\
1 & 2 & & & \\
1 & 6 & 6 & & \\
1 & 14 & 26 & 24 & \\
\vdots & & & \ddots & \\
1 & & & & n!
\end{array}$$

A q-analog for Fubini numbers

- A natural q-analog is $[k]_q! S_{n,k}(q)$, where

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- For example,

$$[3]_q! S_{4,3}(q) = q^5 + 4q^4 + 9q^3 + 11q^2 + 8q + 3.$$

- Recent work (by me and many others) has extended results from $[n]_q!$ to $[k]_q! S_{n,k}(q)$, using...

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- Recent work (by me and many others) has extended results from $[n]_q!$ to $[k]_q! S_{n,k}(q)$, using...
  - ordered set partition statistics,
  - spanning sets for $\mathbb{F}_q^k$, and
  - superspace harmonics.

The statistic minimaj

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Theorem [HRW18, Rho18]

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- Several other equidistributed statistics are studied in [HRW18].

Spanning sets for $\mathbb{F}_q^k$

Theorem

When $n \geq k$, the number of ordered spanning sets

$$(v_1, v_2, \ldots, v_n)$$

for $\mathbb{F}_q^k$ (q a prime power) is

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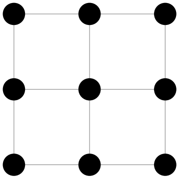
For example, if $q = 3$, $n = 3$, $k = 2$, we get



$$4 \cdot 1 \cdot 3 + 4 \cdot 3 \cdot 4 = 60$$

Superspace

- Let Ω_n denote the space of "polynomials" in two types of variables:
  - $x_1, x_2, \ldots, x_n$, which commute, and
  - $\theta_1, \theta_2, \ldots, \theta_n$, which *anti-commute*, so

$$\theta_i \theta_j = -\theta_j \theta_i \Longrightarrow \theta_i^2 = 0.$$

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- The objects in Ω_n are called *superpolynomials* and appear in mathematical physics and differential algebra [DeW92].

The superspace Vandermonde

- For positive integers $n \geq k$, define the *superspace Vandermonde matrix* to be

$$M_{n,k} = \begin{bmatrix} 1 & x_1 & \dots & x_1^{k-1} & \theta_1 x_1^{k-1} & \dots & \theta_1 x_1^{k-1} \\ 1 & x_2 & \dots & x_2^{k-1} & \theta_2 x_2^{k-1} & \dots & \theta_2 x_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{k-1} & \theta_n x_n^{k-1} & \dots & \theta_n x_n^{k-1} \end{bmatrix}$$

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- Note that $M_{n,n} = M_n$, the usual Vandermonde matrix.
- Define the *superspace Vandermonde determinant* to be

$$\delta_{n,k} = \det\left(M_{n,k}\right)$$

for an appropriate non-commutative determinant.

An example superspace Vandermonde

$$M_{3,2} = \begin{bmatrix} 1 & x_1 & \theta_1 x_1 \\ 1 & x_2 & \theta_2 x_2 \\ 1 & x_3 & \theta_3 x_3 \end{bmatrix}$$

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$$\begin{aligned} \delta_{3,2} &= \det(M_{3,2}) \\ &= \theta_3 x_2 x_3 - \theta_2 x_2 x_3 - \theta_3 x_1 x_3 \\ &+ \theta_1 x_1 x_3 + \theta_2 x_1 x_2 - \theta_1 x_1 x_2 \end{aligned}$$

Superspace harmonics

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- We also explore θ "derivatives," connections to Poincaré duality and the Hard Lefschetz Theorem.

Wrapping up

- Also connections to . . .
  - coinvariants [Zab19],
  - graded dimensions in cohomology [HRS18],
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- Good luck!

Thank you!

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