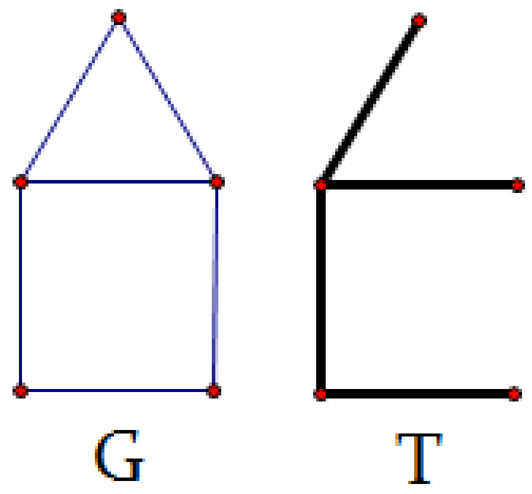
Growing Our Understanding of Tree Graphs

James Mahoney

3-2-15

Spanning Trees

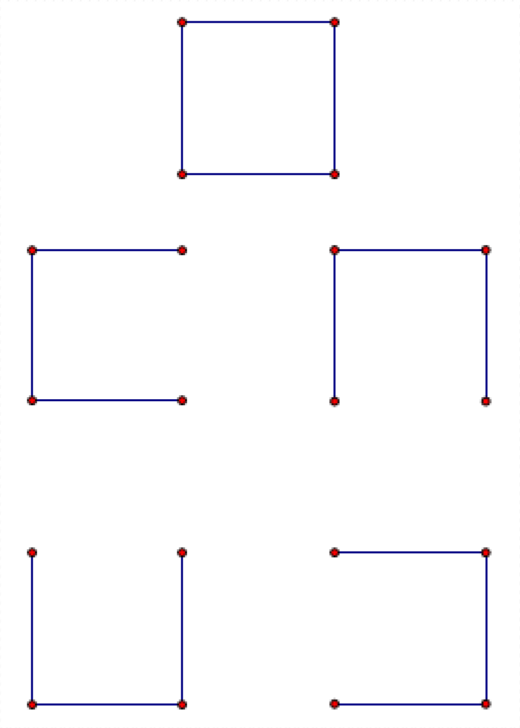
- Papa Bear: Too many edges (contains cycles).
- Mama Bear: Too few edges (some vertices not touched).
- Spanning Tree: Just the right number of edges.
- If a graph has n vertices then a spanning tree will have $n - 1$ edges.



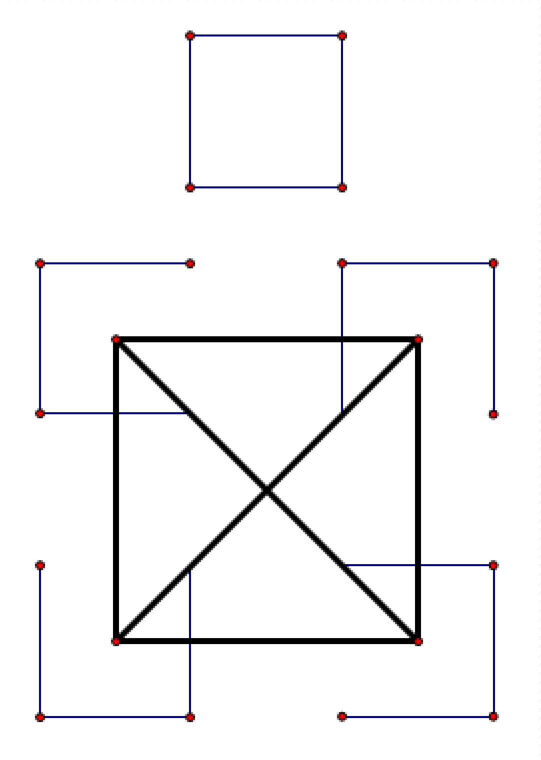
Tree Graphs

- Let G be a connected graph. The *tree graph* of G, $T(G)$, has vertices which are the spanning trees of G, where two vertices are adjacent iff you can change from one to the other by moving exactly one edge.

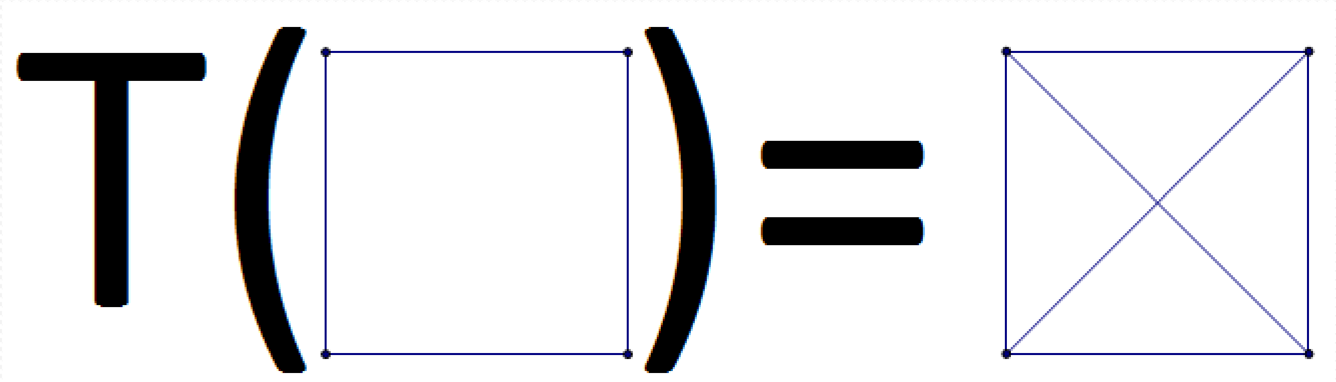
Example: C_4



Example: C_4

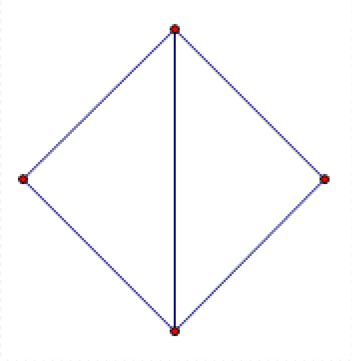


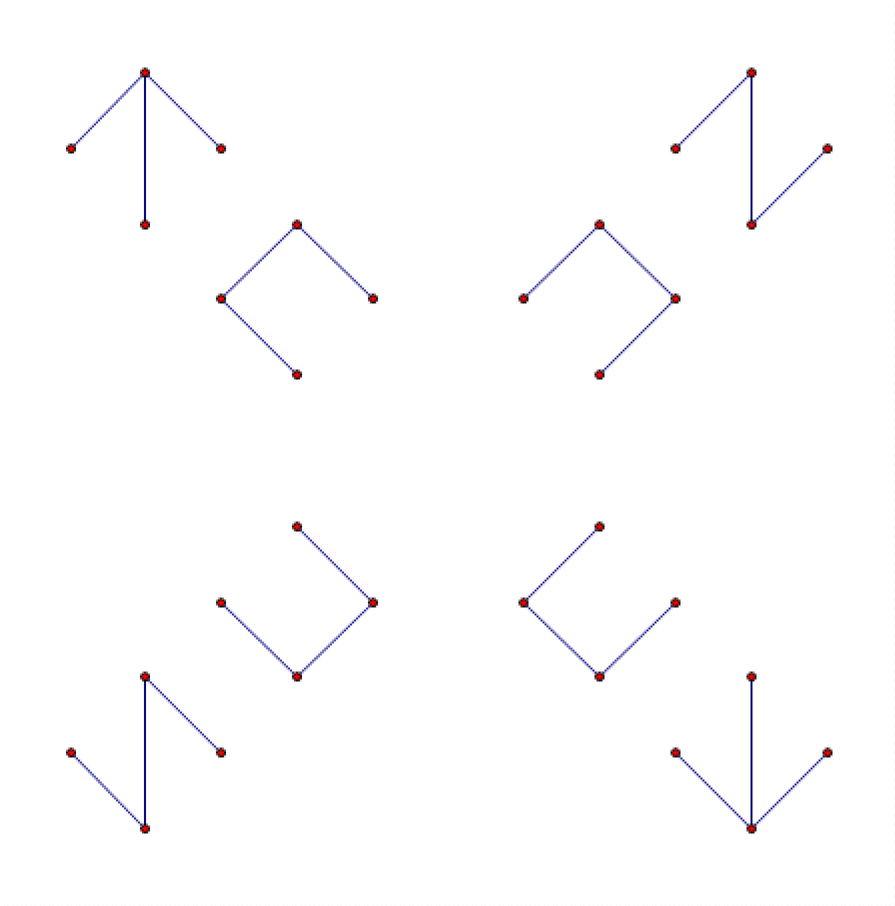
Example: C_4



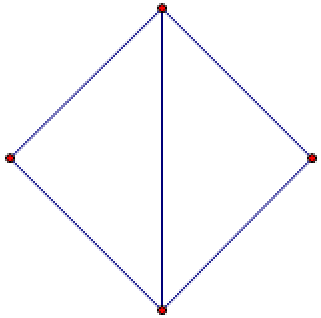
$$T(C_4) = K_4$$

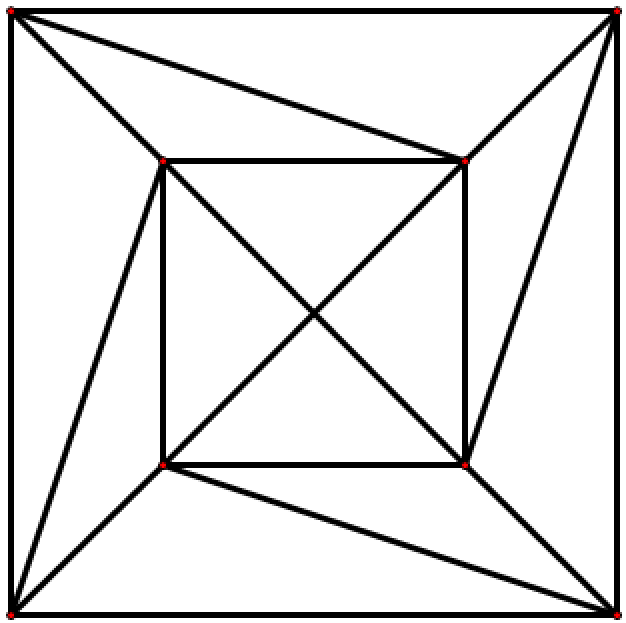
Example: $K_4 - e$





Example: $K_4 - e$





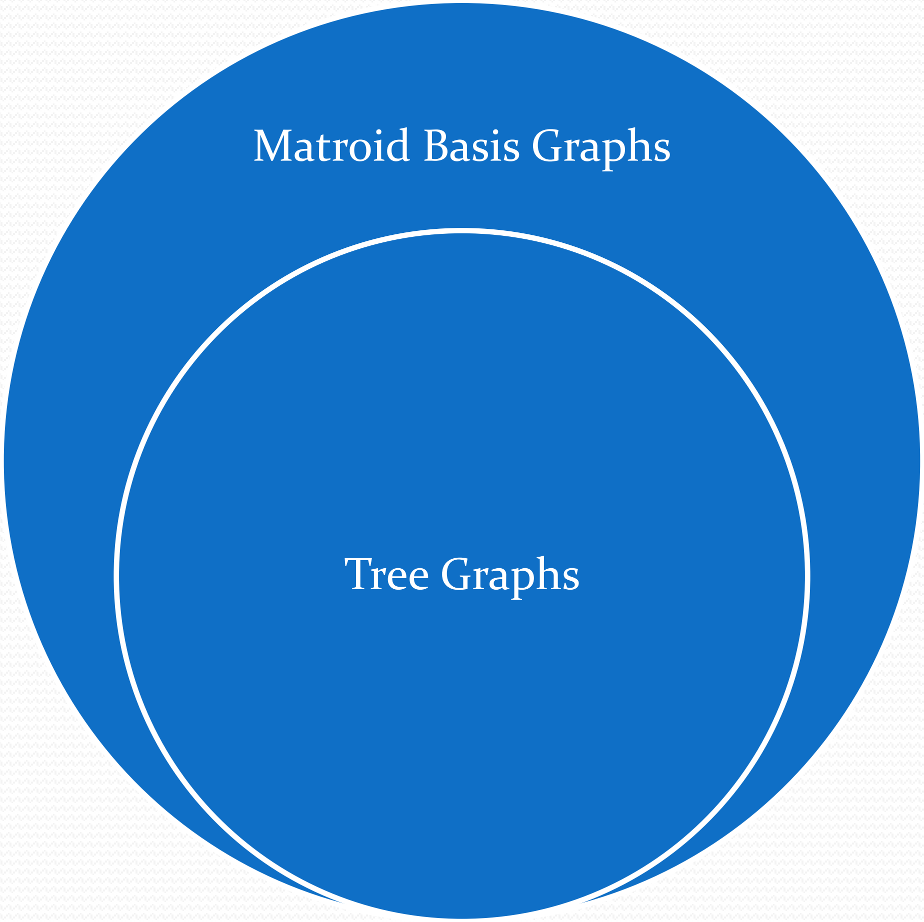
Initial Investigation

- A graph G is *hamiltonian* if there is a cycle that contains every vertex of G.
- For every graph G, is $T(G)$ hamiltonian?
  - Can we move through all of the spanning trees of a graph just by switching one edge at a time?

Known Results

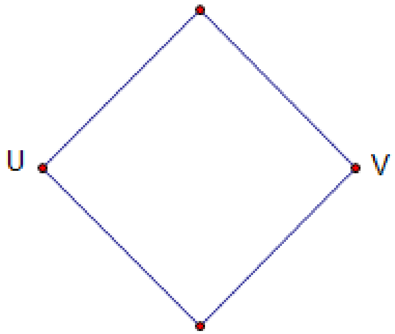
- $T(G)$ is hamiltonian for any graph G (Cummins, 1966)
- $T(G)$ is uniformly hamiltonian (Harary & Holtzmann, 1972)
- $T(G)$ is edge-pancyclic and path-full (Alspach & Liu, 1989)
- $\kappa\big(T(G)\big) = \kappa'\big(T(G)\big) = \delta(T(G))$ (Liu, 1992)
- $\chi\big(T(G)\big) \leq |E(G)|$ (Estivill-Castro, Noy, & Urrutia, 2000)

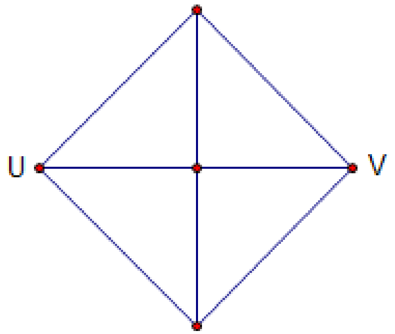
Matroid Basis Graphs



Matroid Basis Graphs

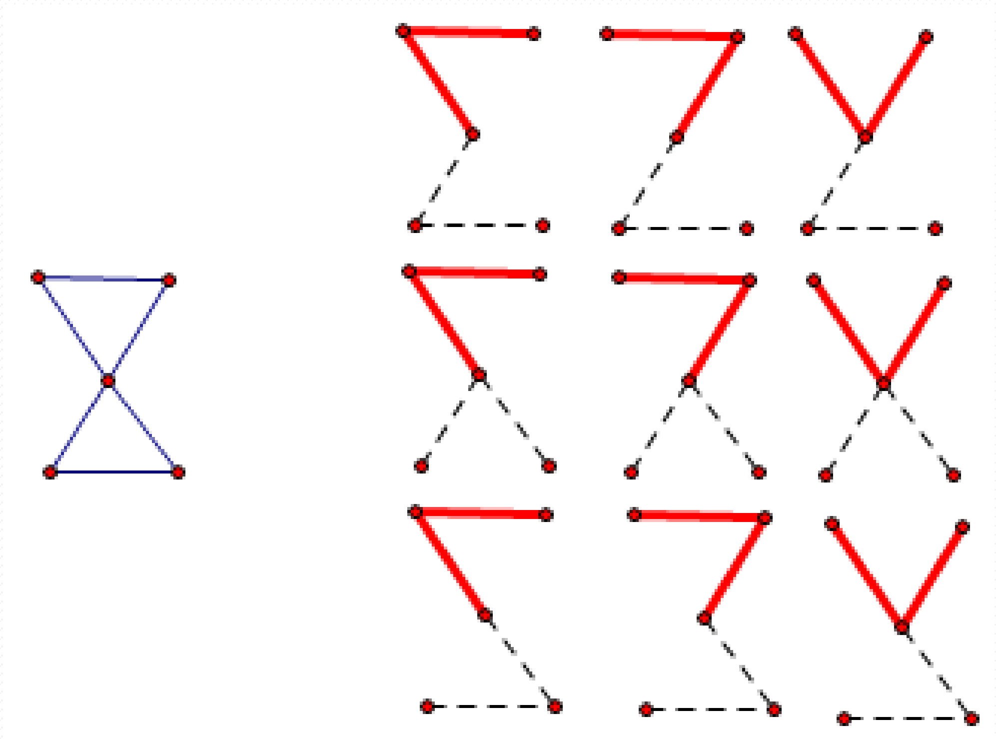
- A *common neighbor subgraph* is the graph induced by two vertices distance two from each other and all of their common neighbors.
- Thm: In a MBG, every common neighbor is either a square, tetrahedron, or octahedron. (Mauer, 1973)
- Thm: Octahedra can't show up in a tree graph, so every common neighbor graph of a tree graph is either a square or a tetrahedron.





Graphs with Cut Vertices

- Let G and H be graphs and let $G \odot H$ be a graph that identifies a vertex in G with a vertex in H.
- Nearem: $T(G \odot H) \cong T(G) \square T(H)$.



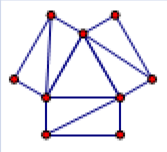
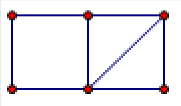
Symmetry of Tree Graphs

- An *automorphism* of a graph G is a permutation of the vertices that respects adjacency. The set of all automorphisms of G forms a group under composition, $Aut(G)$.
- $Aut(G)$ helps describe the symmetries and structure of G.
- The *glory* of a graph G, $g(G)$, is the size of its automorphism group. $g(G) = |Aut(G)|$.
  - If $g(G)$ is large, G is highly symmetric (glorious).

$Aut(T(G))$

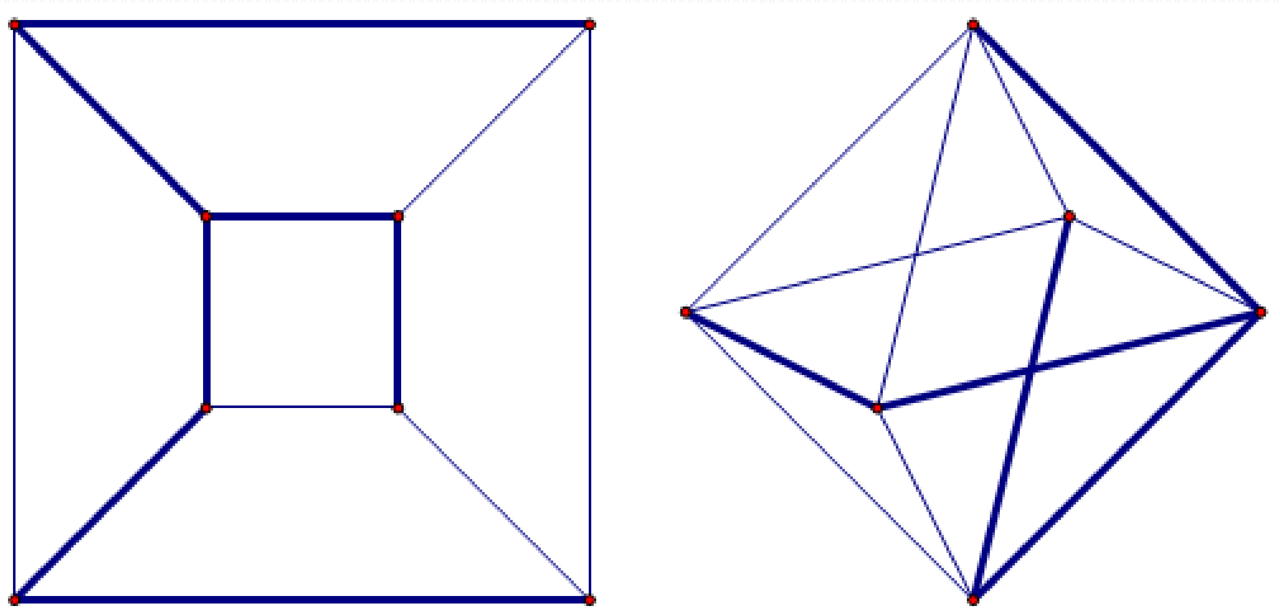
- $g(G)$ has been large for most of the small graphs studied so far.
- Thm: If G is 3-connected, then $Aut(G) \cong Aut(T(G))$.
- Conj: The converse is true as well.
- Conj: If G is 2-connected but not 3-connected, $g(T(G))$ is either 1, 2, 6, or is divisible by 4.
- Nearem: $Aut(G)$ is a subgroup of $Aut(T(G))$.
  - $Aut(K_4 - e) \cong V_4$ while $Aut\big(T(K_4 - e)\big) \cong D_8$, the symmetries of the square.
- Conj: $g(T(G))/g(G)$ is 1, 3, or even.

Automorphism Size Examples

Graph G	$g(T(G))$	$g(G)$	Notes
$K_4 - e$	8	4	D_8 and V_4
$K_{3,2}$	48	12	$S_4 \times S_2$ and $S_3 \times S_2$
K_5	120	120	Probably S_5 and S_5
$C_6 \ominus C_6$	28800	4	? and V_4
	288	3	? and $\mathbb{Z}_3$
	12	1	D_{12} and trivial
C_4	24	8	S_4 and D_8

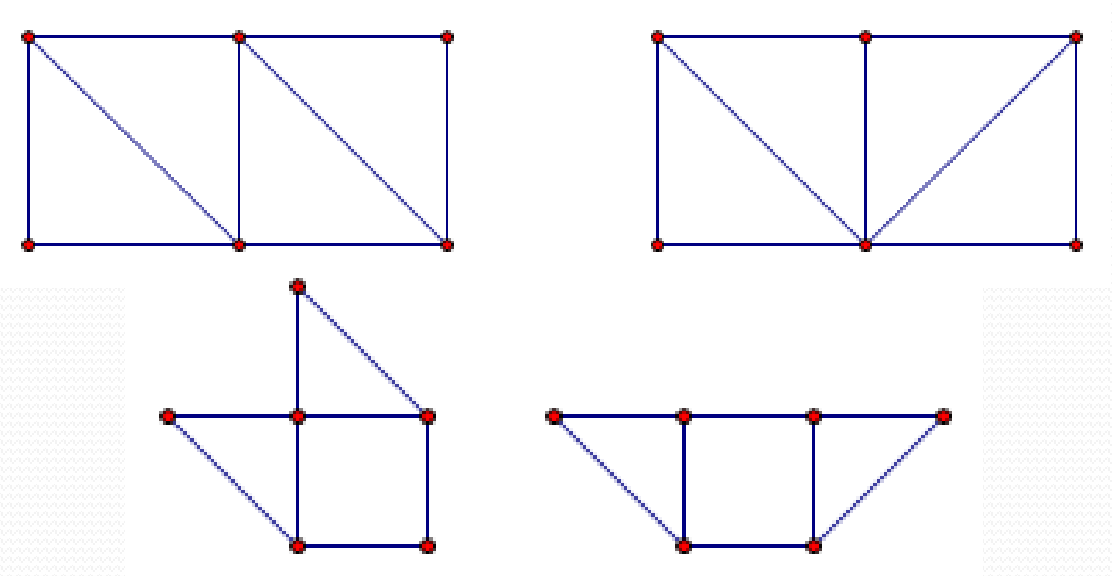
Isomorphic Tree Graphs

- Is it ever the case that $G \ncong H$ but $T(G) \cong T(H)$?
- Thm: If G is 3-connected and planar, $T(G) \cong T(G^*)$. Planar duals give isomorphic tree graphs.
  - Halin graphs and polyhedral graphs fit this.



Isomorphic Tree Graphs

- These pairs of graphs are not isomorphic, but their tree graphs are.
- The starting graphs are *isoparic*: they have the same number of vertices and same number of edges but are not isomorphic.

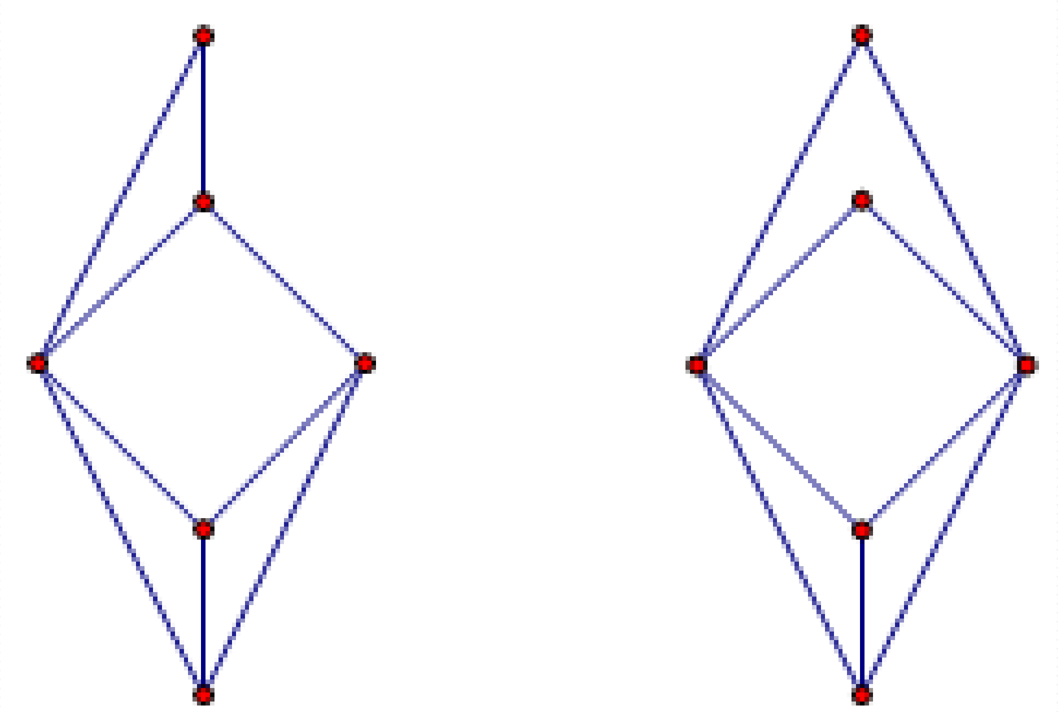


Realizing Tree Graphs

- Given $T(G)$, can we find a graph H such that $T(H) \cong T(G)$?
- Given the number of vertices and edges in a tree graph, can we put a useful bound on the number of vertices or edges in the original graph?

Realizing Tree Graphs

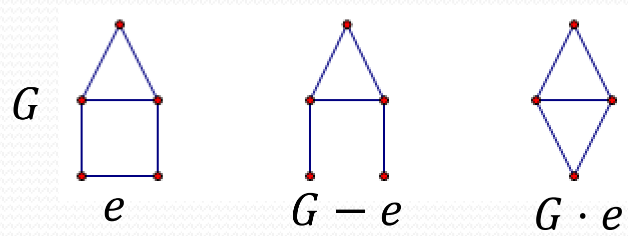
- These two graphs are isoparic and their tree graphs are isoparic (both have 64 vertices and 368 edges).

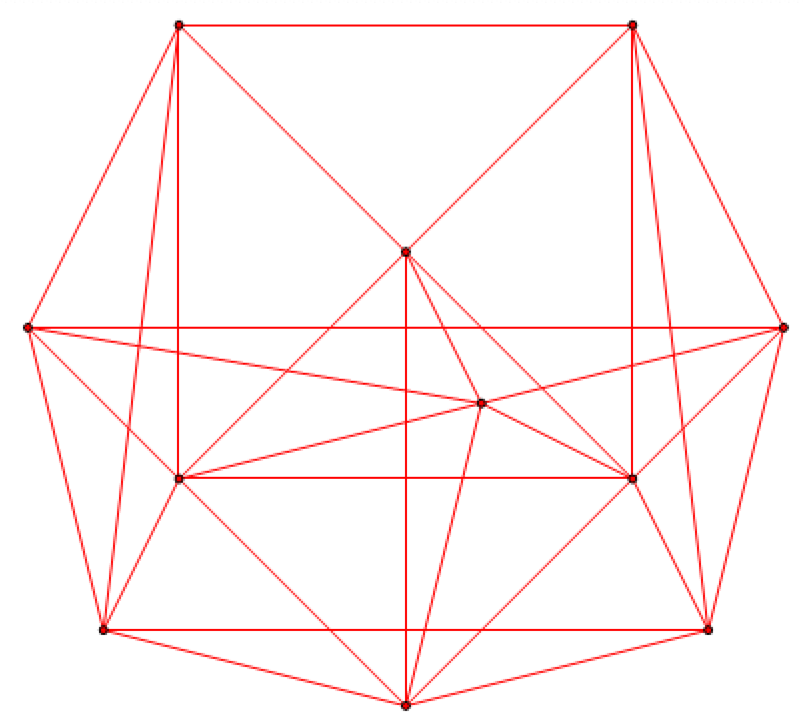


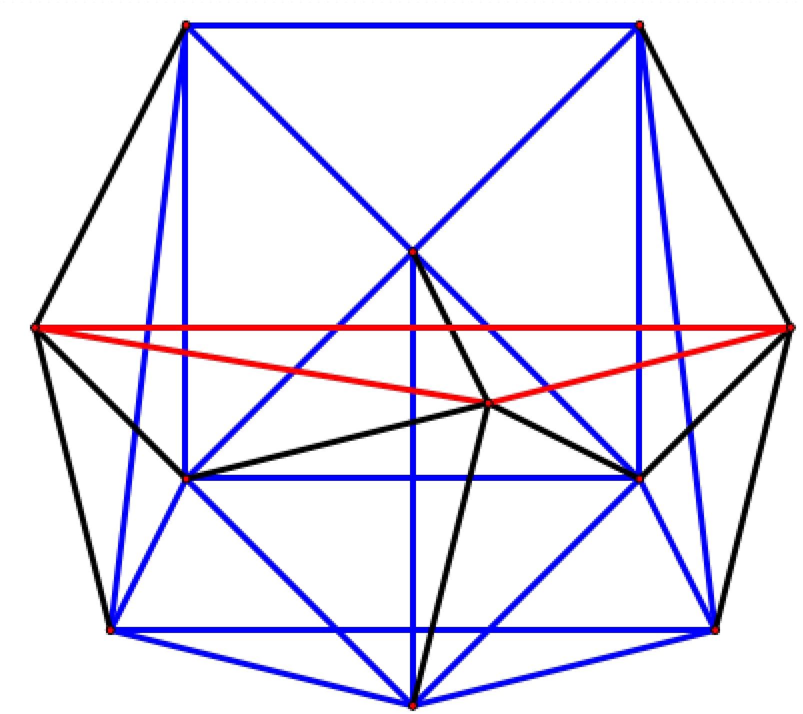
Deletion/Contraction

- Let e be an edge of G.
- Nearem: $T(G) \cong T(G - e) \cup T(G \cdot e) \cup$ additional edges.

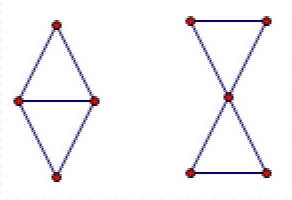
Deletion/Contraction



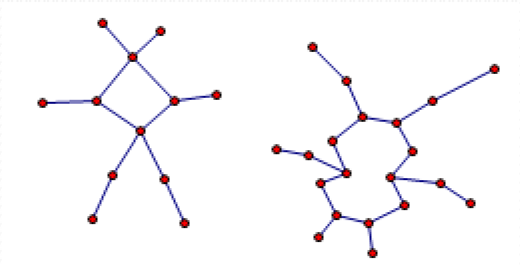




Planarity

Diamond  Butterfly

- Thm: If the both the diamond and the butterfly are forbidden minors, the family of graphs obtained are the pseudoforests.
- Pseudoforest: every connected component is a unicycle.
- Unicycle: a connected graph with exactly one cycle.
- Thm: The tree graphs of the diamond and the butterfly are nonplanar. (Contain K_5 and $K_{3,3}$ minors, respectively.)
- Nearem: $T(G)$ is nonplanar unless $G \cong C_3, C_4$.

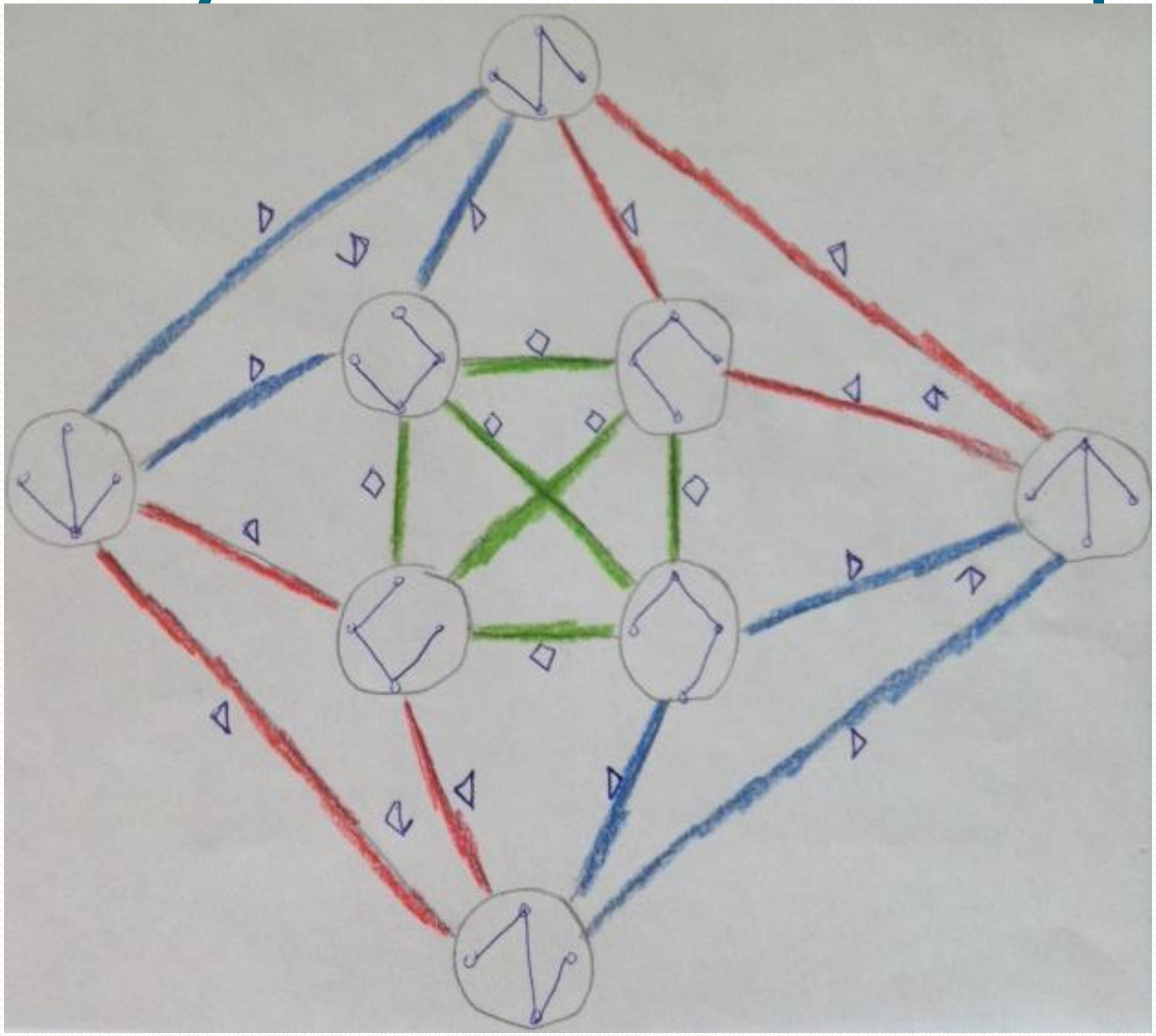


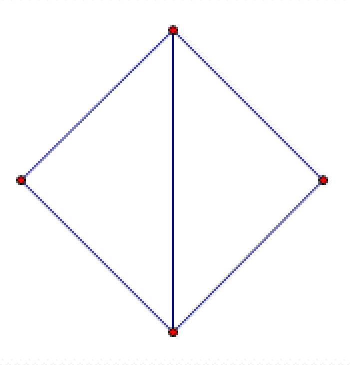
Pseudoforest made of unicycles

Unicycles and Decomposition

- Each spanning tree has $n - 1$ edges. G has m edges. To find the neighbors of a vertex in $T(G)$, add one of the $m - (n - 1) = m - n + 1$ "extra" edges to the tree. Exactly one cycle will be formed.
- This unicycle of size c will give rise to a K_c subgraph in $T(G)$.
- Nearem: The edges of $T(G)$ can be decomposed into cliques of size at least three such that each vertex is in exactly $m - n + 1$ cliques.
  - Can be used to predict number of edges in $T(G)$.
  - Is this decomposition unique?

Unicycles and Decomposition





$m = 5$
$n = 4$
$m - n + 1 = 2$

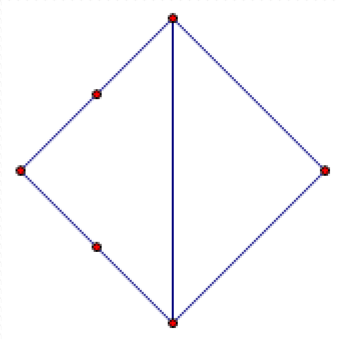
Degree Bounds

- Let $v = m - n + 1$.
- Thm: $[girth(G) - 1][v] \leq \delta\big(T(G)\big)$ $\leq \Delta(T(G)) \leq [circ(G) - 1][v]$ (Liu, 2002)
- By Vizing's Theorem we then have

$$[girth(G) - 1][v] \leq \chi'\big(T(G)\big) \leq [circ(G) - 1][v] + 1$$

Additional Families

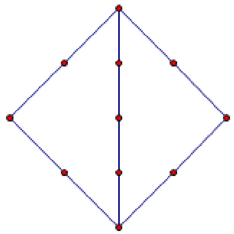
- Let $\theta_{l,m,n}$ be the graph with two vertices joined by three disjoint paths of edge length l, m, and n.
- Thm: $T(\theta_{l,m,n}) \cong L(K_{l,m,n})$.
- Nearem: This is the only time a tree graph will be a line graph.



$\theta_{4,2,1}$

Additional Families

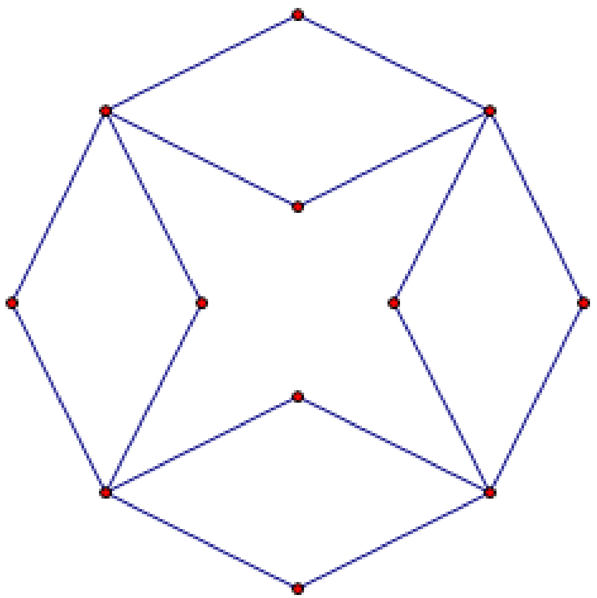
$P_{3,4}$

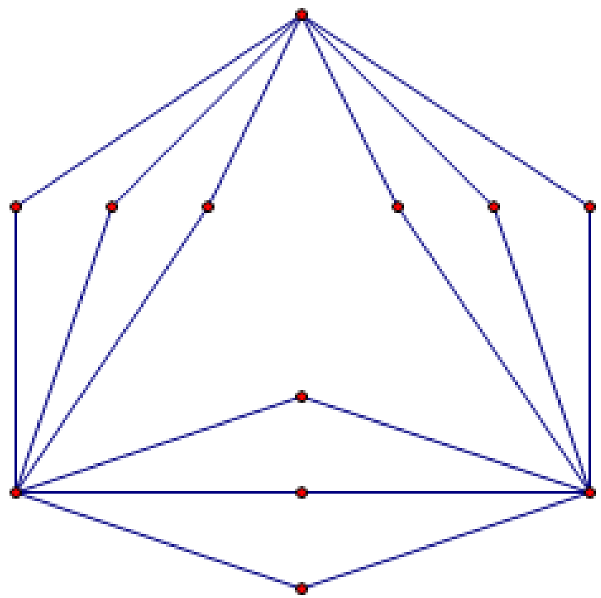


- Let $P_{n,k}$ be the graph where two vertices are joined by n disjoint paths of edge length k.
- Thm: $T(P_{n,k})$ is $(n-1)(2k-1)$-regular.
  - Any number that is not a power of 2 can be written in this form.
- Conj: $T(P_{n,k})$ is integral (with easily-understood eigenvalues) and vertex transitive.
- $T(P_{n,k})$ could be a new infinite family (with two parameters) of regular integral graphs.

Additional Families

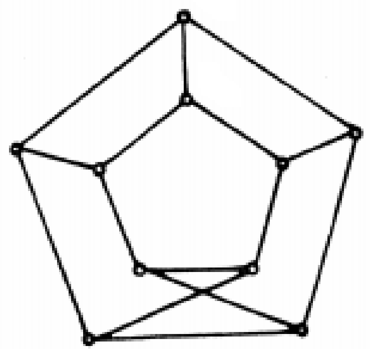
- Bracelets: Take a graph and join together copies of it like a bracelet.
- Conj: If the repeated graph has a regular tree graph, the tree graph of the bracelet will be regular.





Additional Families

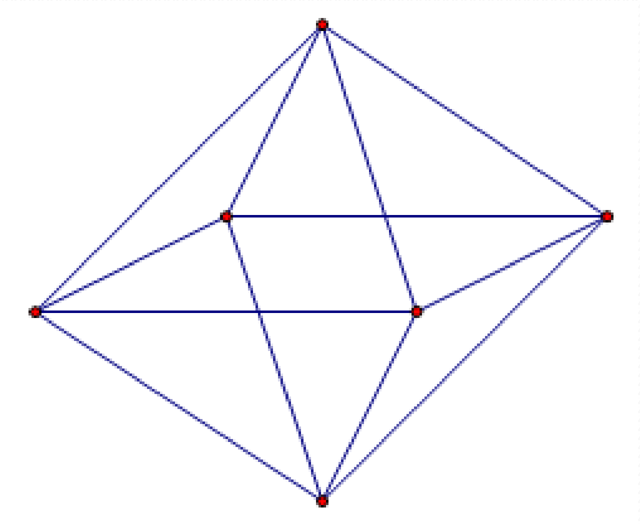
- So far cycles, $P_{n,k}$, and bracelets have been the only families that induce regular tree graphs. While $T(P_{n,k})$ seems to be integral, that doesn't seem to hold for bracelets.
- The tree graphs of these families seem to be vertex-transitive.
- Conj: If $T(G)$ is regular then it is vertex-transitive.

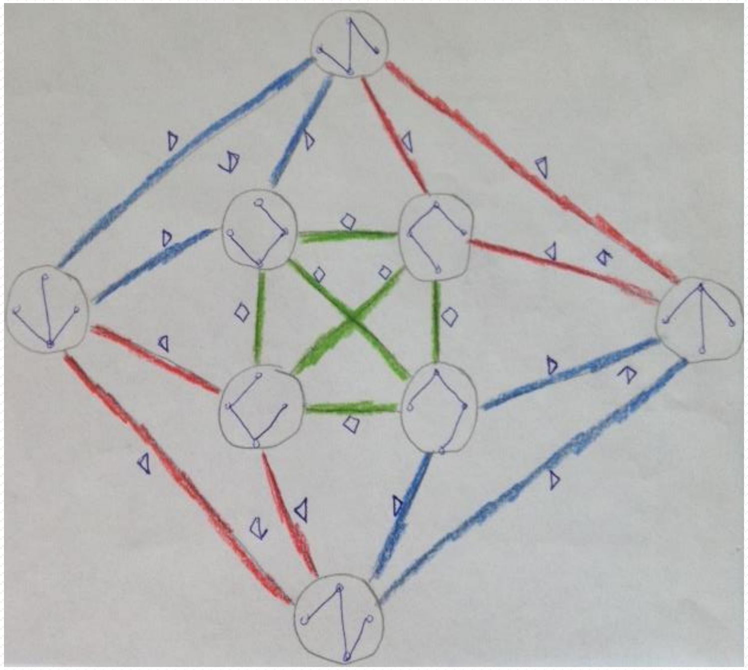


(Zelinka, 1990)

Characterizing Tree Graphs

- Conj: $H \cong T(G)$ iff it is a matroid basis graph with no induced octahedra and can be decomposed into cliques of size 3 or more, where each vertex is in the same number of cliques.



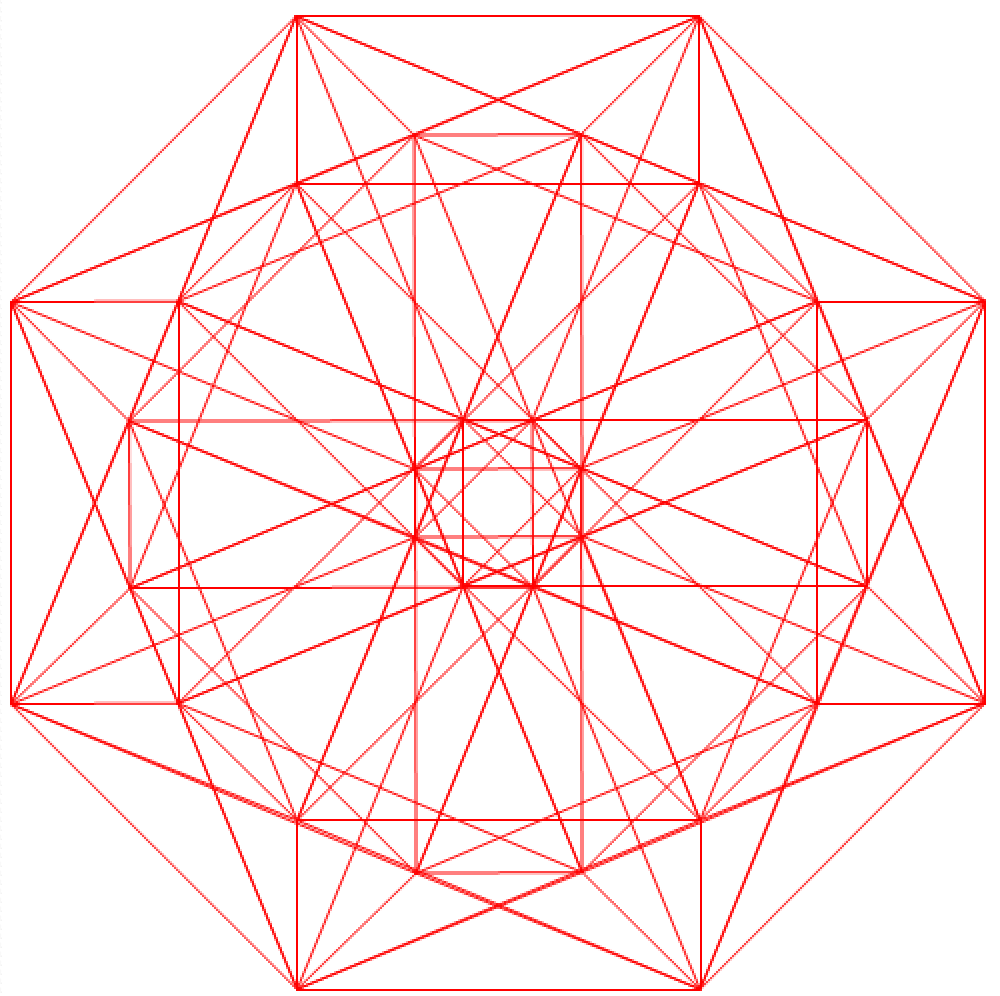


Further Research

- Can we find good bounds for other graph parameters such as ω and α?
- Can we find better bounds for χ and Δ?
- Can tree graphs be characterized in a simple way?
- Does the number of cycles or cycle-types in G affect the regularity or integrality of $T(G)$? If so, how?
- Are there other families of regular or integral tree graphs?

Thanks!

- Any questions?



$T(K_{4,2})$