The Strong Law of Small Numbers

James Mahoney

How much is enough?

- 1, 2, 3, 4, 5:
- 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, … (natural numbers)
- 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, … (deficient numbers)
- 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, … (prime powers)
- 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, … (narcissistic numbers)

The Strong Law of Small Numbers

- "There aren't enough small numbers to meet the many demands made of them."
  - "Capricious coincidences cause careless conjectures."

    vs.
  - "Initial irregularities inhibit incisive intuition."

You decide

- Unlike patterns such as the Collatz conjecture, all of these are proven to be either genuine or coincidental
- Your job is to decide, for each example, what category it falls into

Pattern 1

- $2^{2^0} + 1 = 3$
- $2^{2^1} + 1 = 5$
- $2^{2^2} + 1 = 17$
- $2^{2^3} + 1 = 257$
- $2^{2^4} + 1 = 65537$

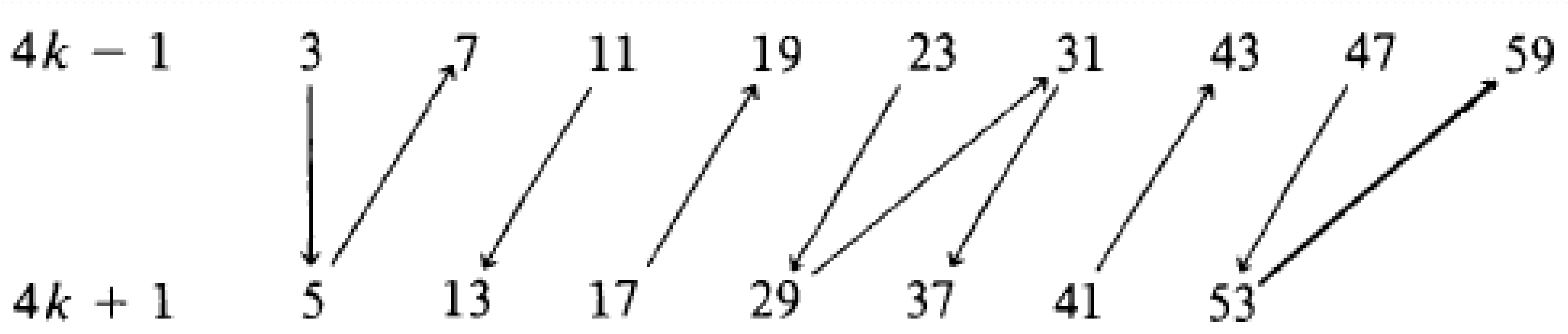
All of which are prime

Pattern 2

- $2^2 - 1 = 3$
- $2^3 - 1 = 7$
- $2^5 - 1 = 31$
- $2^7 - 1 = 127$

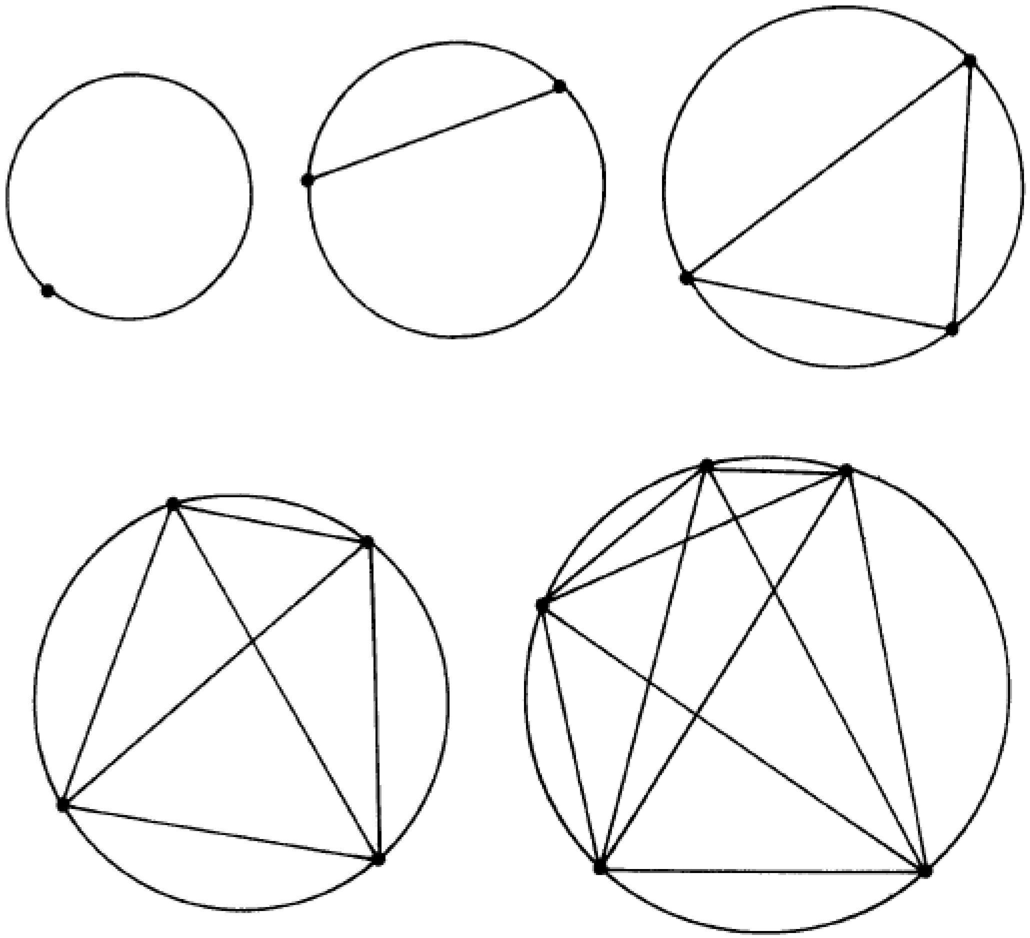
All of which are prime

Pattern 3



- Claim: At least as many primes of form $4k - 1$ as form $4k + 1$ as we go up through the natural numbers

Pattern 4



- How many regions?
- 1, 2, 4, 8, 16

Pattern 5

- 31
- 331
- 3331
- 33331
- 333331
- 3333331

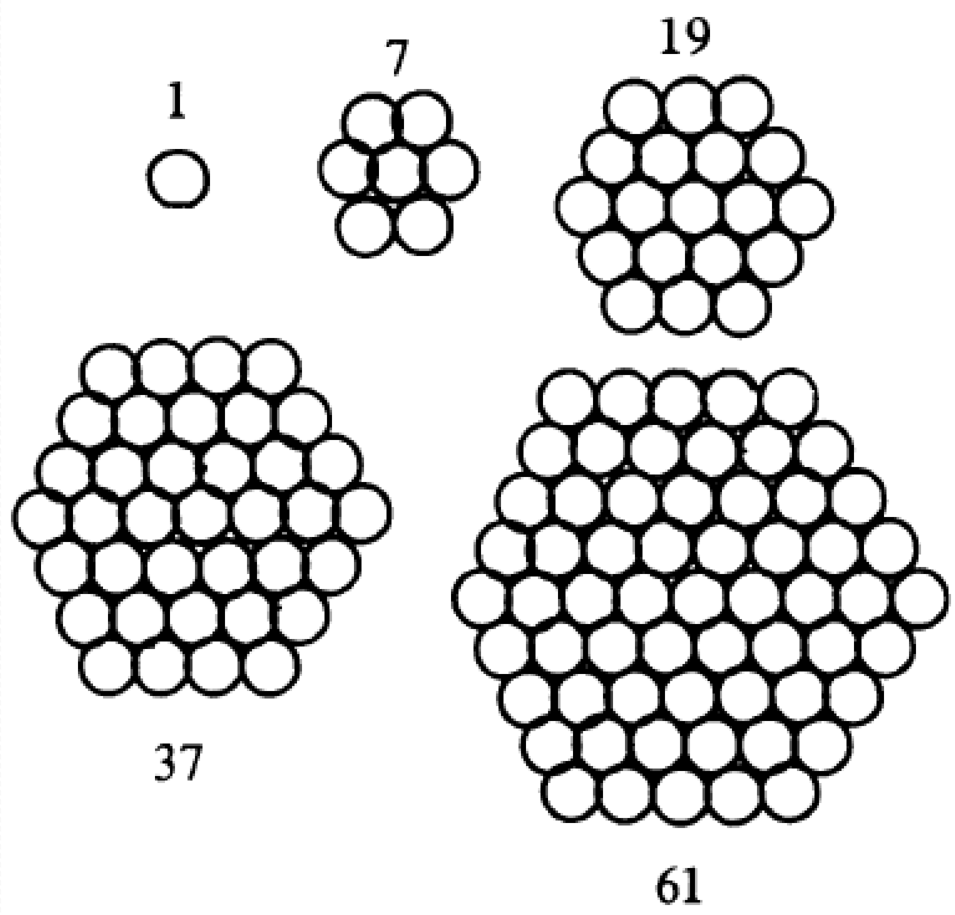
All are prime.

Pattern 6

$$3! - 2! + 1! = 5$$
$$4! - 3! + 2! - 1! = 19$$
$$5! - 4! + 3! - 2! + 1! = 101$$
$$6! - 5! + 4! - 3! + 2! - 1! = 619$$
$$7! - 6! + 5! - 4! + 3! - 2! + 1! = 4421$$
$$8! - 7! + 6! - 5! + 4! - 3! + 2! - 1! = 35899$$

All are prime.

Pattern 7



- 1
- $1 + 7 = 8$
- $1 + 7 + 19 = 27$
- $1 + 7 + 19 + 37 = 64$
- $1 + 7 + 19 + 37 + 61 = 125$

Pattern 8

1	~~2~~	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11
1		4		9		16		25		36

- Cross off every other natural number then take partial sums. They are all perfect squares.

Pattern 9

1	2	~~3~~	4	5	~~6~~	7	8	~~9~~	10	11	~~12~~	13	14	~~15~~	16
1	~~3~~		7	~~12~~		19	~~27~~		37	~~48~~		61	~~75~~		91
1			8			27			64			125			216

- Delete every third, take partial sums, delete every second, then take partial sums. They are all perfect cubes.

Pattern 10

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369
1				16				81				256				625

- Delete every fourth, take partial sums, delete every third, take partial sums, delete every second, take partial sums. They are all fourth powers.

Pattern 11

① 2 ~~3~~ 4 5 ~~6~~ 7 8 9 ~~10~~ 11 12 13 14 ~~15~~ 16 17 18 19 20 ~~21~~

② 6 ~~11~~ 18 26 ~~35~~ 46 58 71 ~~85~~ 101 118 136 155 ~~175~~

⑥ 24 ~~50~~ 96 154 ~~225~~ 326 444 580 ~~735~~

(24) 120 ~~274~~ 600 1044 ~~1624~~

(120) 720 ~~1764~~

(720)

Circle the first number in the row. Skip one, delete, skip two, delete, skip three, delete, etc. Take partial sums. Repeat. They are all factorials.

Pattern 12

- Let

$$f(x) = \frac{x^4 - 6x^3 + 23x^2 - 18x + 24}{24}$$

$f(1) = 1$
$f(2) = 2$
$f(3) = 4$
$f(4) = 8$
$f(5) = 16$

Pattern 13

	0	1	2	3	4	5	6	7	8	9
0	1									
1		1	1							
2			1	2	1					
3				1	3	3	1			
4					1	4	6	4	1	
5						1	5	10	10	5
6							1	6	15	20
7								1	7	21
8									1	8
9										1
	1	1	2	3	5	8	13	21	34	55

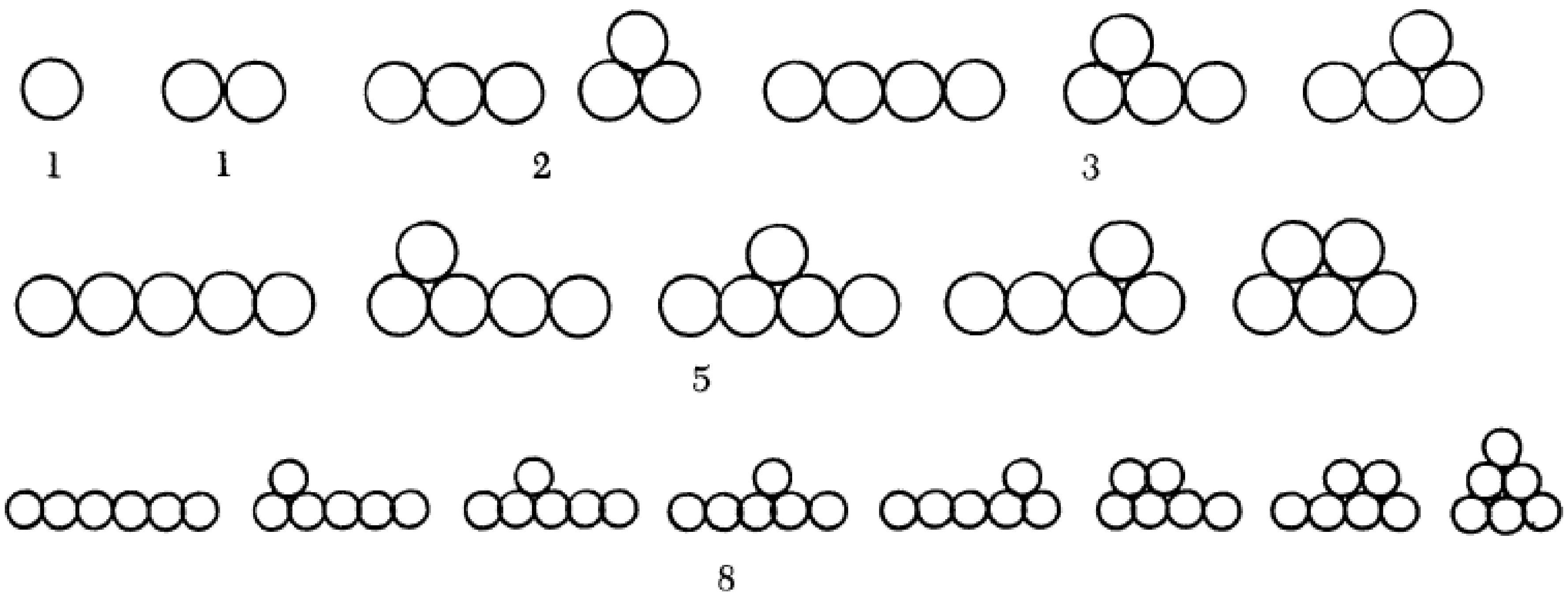
- Offset rows of Pascal's triangle and sum the columns. The answers are all Fibonacci numbers.

Pattern 14

0	1	2	3	4	5	6	7	8	9
1	1	2	3	5	8	13	21	34	55

- The function $\left\lceil e^{\frac{x-1}{2}} \right\rceil$ gives the Fibonacci numbers.

Pattern 15



- Stack pennies in rows such that each row has no gaps between pennies and pennies in upper rows touch two pennies beneath them. The number of ways to do this gives the Fibonacci numbers.

Pattern 16

- Let $f(x) = 9x^2 - 231x + 1523$

$$f(0) = 1523$$
$$f(1) = 1301$$
$$f(2) = 1097$$
$$f(3) = 911$$
$$f(4) = 743$$
$$f(5) = 593$$
$$f(6) = 461$$

These are all prime numbers.

Pattern 17

- Let $f(x) = \lfloor 1.5^x \rfloor$

$$f(2) = 2$$
$$f(3) = 3$$
$$f(4) = 5$$
$$f(5) = 7$$
$$f(6) = 11$$
$$f(7) = 17$$

These are all prime.

Pattern 18

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\,dx = \pi/2$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\,dx = \pi/2$$

Answer 1

- Claim: $2^{2^n} + 1$ always results in a prime
- FALSE
- These are the Fermat primes. He erroneously made this claim. Euler showed that $2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417$
- It is not known if this set is finite.

Answer 2

- Claim: $2^p - 1$ is prime when p is prime
- FALSE
- These are the Mersenne primes. $2^{11} - 1 = 23 \cdot 89$.
- It is not known if this set is finite.

Answer 3

- Claim: At least as many primes of form $4k - 1$ as form $4k + 1$ as we go up through the natural numbers
- FALSE
- Each form takes the lead infinitely often (Littlewood, 1914). First reversal is at $n = 26861$.

Answer 4

- Claim: Number of regions made by drawing all chords between n points on a circle is 2^{n-1}.
- FALSE

# of points =	1	2	3	4	5	6	7	8	9	10
# of regions =	**1**	**2**	**4**	**8**	**16**	31	57	99	163	**256**

$$\binom{n-1}{4} + \binom{n-1}{3} + \binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0}$$

Answer 5

- Claim: The number $3\ldots31$ is always prime
- FALSE
- $333333331 = 17 \cdot 19607843$

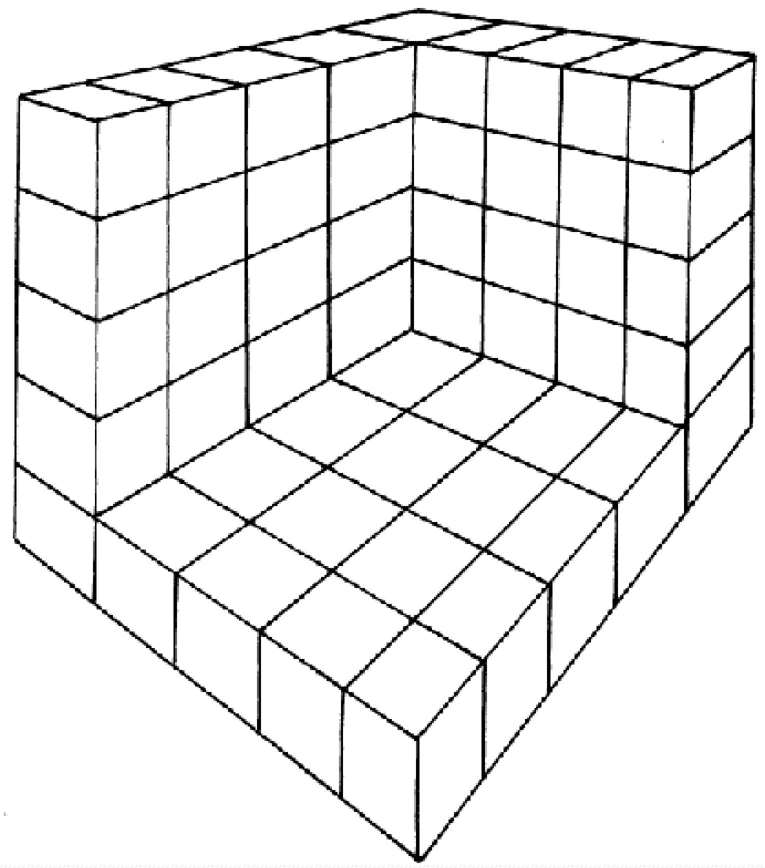
Answer 6

- Claim: Alternating sums of factorials are always prime.
- FALSE

$$9! - 8! + 7! - 6! + 5! - 4! + 3! - 2! + 1!$$
$$= 326981 = 79 \times 4139$$

Answer 7

- Claim: Partial sums of hex numbers are perfect cubes.
- TRUE



Answers 8-11

- Claim: Deleting patterns of natural numbers and taking partial sums can produce powers and factorials.
- TRUE
- These techniques are called *Moessner's Process* after Moessner, who first conjectured these findings. Later proven by Perron and generalized by other mathematicians.

Answer 12

- Claim: There is a polynomial that interpolates all of the powers of 2.
- FALSE
- This is in fact the polynomial given by

$$\binom{n-1}{4}+\binom{n-1}{3}+\binom{n-1}{2}+\binom{n-1}{1}+\binom{n-1}{0}$$

which is from the circle chord region problem.

Answer 13

- Claim: Fibonacci numbers can be found in Pascal's triangle.
- TRUE
- There are some nice combinatorial proofs of this.

Answer 14

- Claim: The function $\left\lceil e^{\frac{x-1}{2}} \right\rceil$ gives the Fibonacci numbers.
- FALSE
- Plugging in the next value, 10, gives 91 instead of 89. The spread gets worse from there.

Answer 15

- Claim: Number of ways to arrange penny stacks gives the Fibonacci numbers.
- FALSE
- The number of arrangements for 7, 8, and 9 pennies is 12, 18, and 26, respectively.

Answer 16

- Claim: There exists a polynomial that only outputs primes on nonnegative integer inputs.
- FALSE
- Goldbach (1792) showed that no such polynomial can exist (just plug in the constant value).
- The given polynomial is prime for 0 to 39.
- The current record is 57 distinct primes in a row.

Answer 17

- Claim: $f(x) = \lfloor 1.5^x \rfloor$ produces only prime outputs from 2 on.
- FALSE
- $f(8) = 25$
- $f(x)$ is not prime again until $f(21) = 4987$
- Mills (1947) proved the existence of a constant A such that $\lfloor A^{3^n} \rfloor$ is prime for all positive integers n.

Answer 18

- Claim: integrating appropriately patterned products of the sinc function from 0 to ∞ always gives $\pi/_2$.
- FALSE
- This is known as the *Borwein Integral*. The pattern actually continues on until the $\frac{1}{13}$ term. At $\frac{1}{15}$ the value becomes $\frac{\pi}{2} - \frac{6879714958723010531}{935615849440640907310521750000}\pi$

  This pattern will hold for any sequence of positive real numbers whose sum is less than 1
- $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \approx .9551$
- $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} \approx 1.0218$

How did you do?

References

- All material is from Richard Guy's papers
  The Strong Law of Small Numbers and
  The Second Strong Law of Small Numbers,
  except for the Borwein Integral material, which came from Wikipedia.

THANK YOU!

- Any questions?