# Here’s Rookin’ at You, Kid

## Rook Polynomials as a Unifying Combinatorial Concept

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# A **<u>rook</u>** is a …

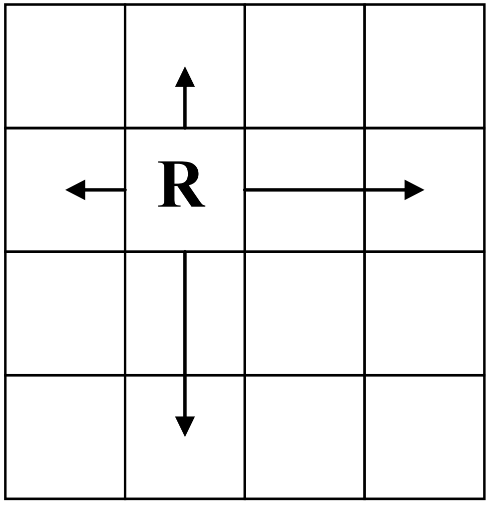
a. ~~bird.~~

b. ~~card game.~~

c. chess piece.

d. ~~swindler or cheat.~~

e. ~~all of the above.~~



## A **rook** is a …

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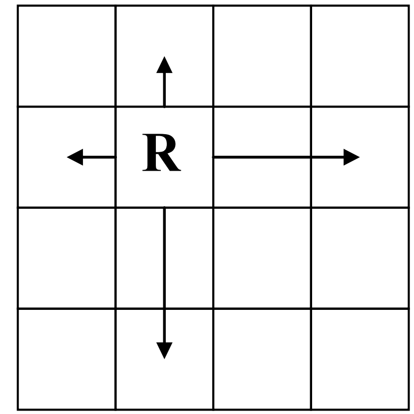
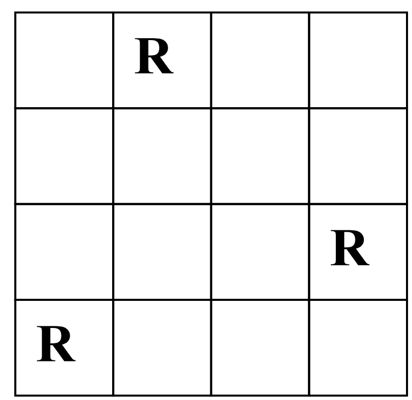


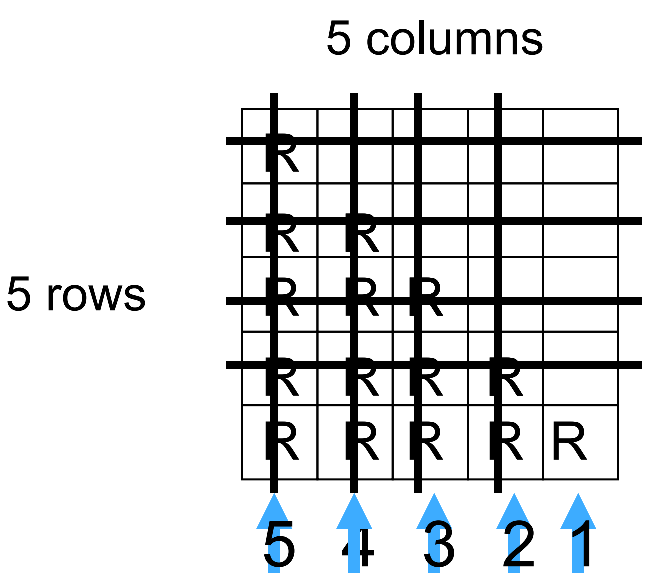

– We want to count "non-attacking" configurations of rooks.

# Rook Boards 101

How many ways can we place $r$ non-attacking rooks on an $m$ x $n$ chessboard?

# Rook Boards 101

- Count the ways to place $r$ non-attacking rooks on an $m$ x $n$ board:
  - A very special case: $r$ rooks on an $r$ x $r$ board
    - Example: a 5 x 5 board




# of ways to place 5 rooks on a 5x5 board $= 5 \times 4 \times 3 \times 2 \times 1$

$= \mathbf{5!}$

# Rook Boards 101

- Count the ways to place *r* non-attacking rooks on an *m* x *n* board:
  - The # of ways to place *r* rooks on an *r* x *r* board is *r* !
  - Recall "choose" notation, where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

is the number of ways of choosing *r* objects from *n* distinct objects

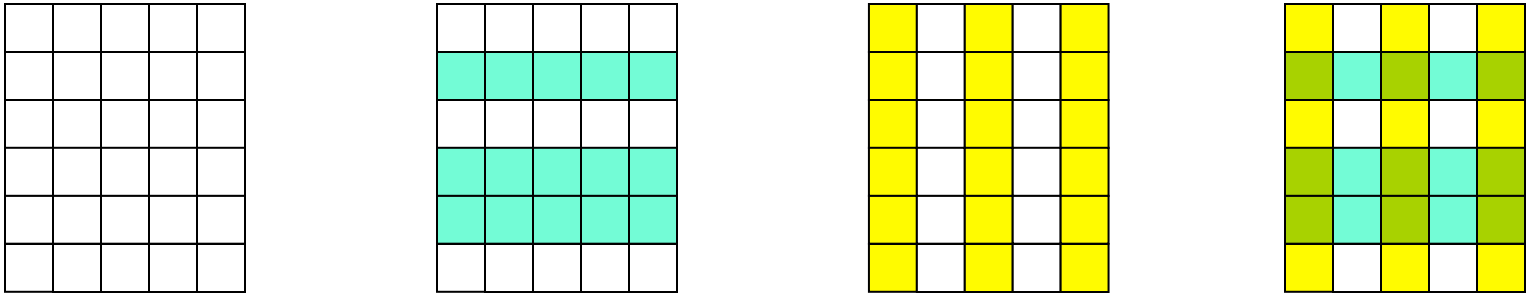
# Rook Boards 101

- Count the ways to place $r$ non-attacking rooks on an $m$ x $n$ board:
  - The # of ways to place $r$ rooks on an $r$ x $r$ board is $r$ !
  - Recall "choose" notation, $\binom{n}{r}$
- So, to count the ways to place $r$ rooks on an $m$ x $n$ board, we
  - Choose $r$ rows
  - Choose $r$ columns
  - Place the rooks among these $r$ x $r$ squares

# of non-attacking configurations of $r$ rooks on an $m$ x $n$ board $= \binom{m}{r}\binom{n}{r} r!$

# Rook Boards 101

- Example
  - How many ways are there to place 3 rooks on a 6 x 5 board?



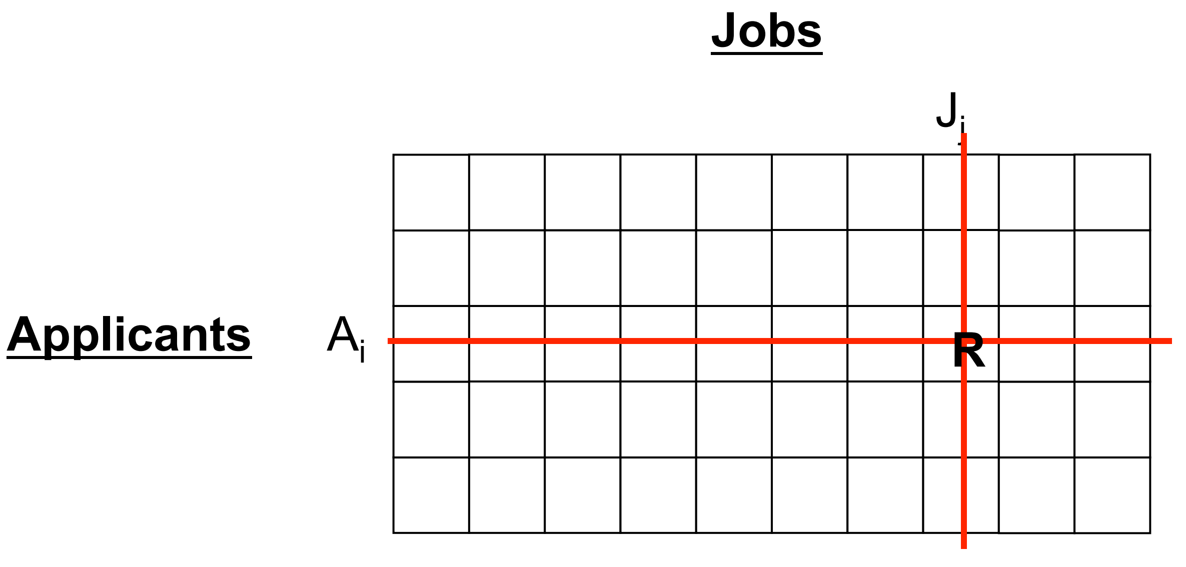
$$\binom{m}{r}\binom{n}{r}r! = \binom{6}{3}\binom{5}{3}3! = 20\cdot 10\cdot 6 = 1200$$

# Rooks in the Real World

- There are 10 companies each hiring one position, and there are 5 applicants applying to these 10 jobs.
- Count the number of different ways that these jobs could be filled.

# Rooks in the Real World

- How does this relate to non-attacking rooks?
- What does a rook mean in the jobs context?
- Why non-attacking rooks?

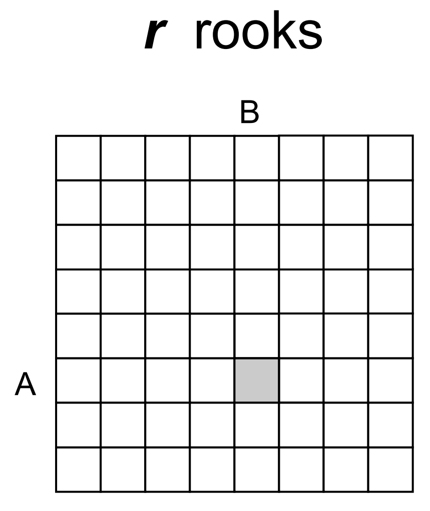


# Taboo Squares

- One of the jobs is as a bartender.
- Suppose Amy is ineligible to work at a bar because she’s too young.
    - Such a constraint is a **restricted position**
    - Introduces **Inclusion/Exclusion Principle**

# Taboo Squares

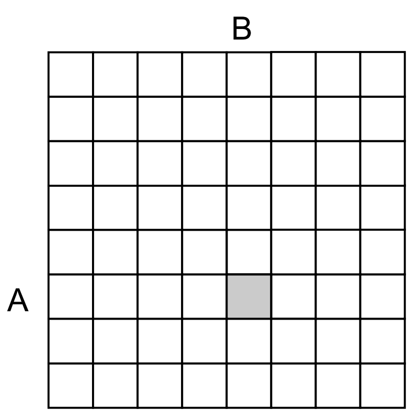
- How to account for one restricted position



# Taboo Squares

- How to account for one restricted position
- Idea: "Total – Bad"

***r*** rooks

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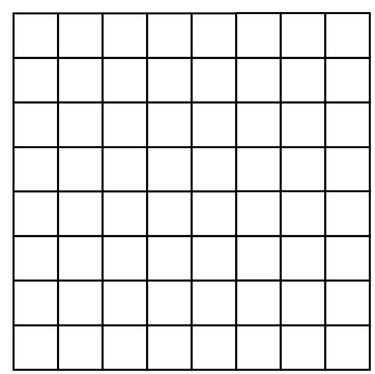
***r*** rooks



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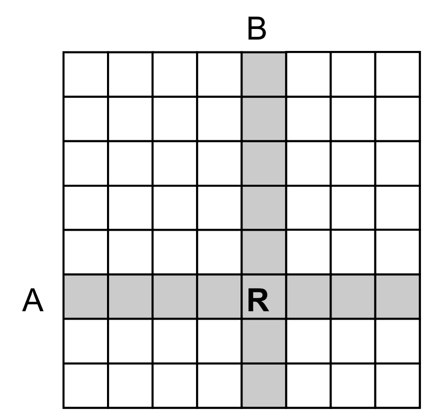
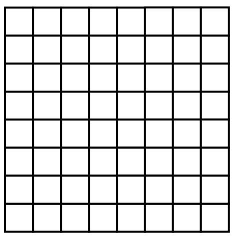
***r*** - 1 rooks




$$\binom{m}{r}\binom{n}{r}r! \quad - \quad \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

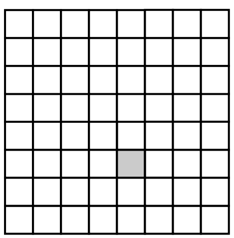
# Taboo Squares

- m x n board → no restricted positions



$$\binom{m}{r}\binom{n}{r}r!$$

- m x n board → one restricted position

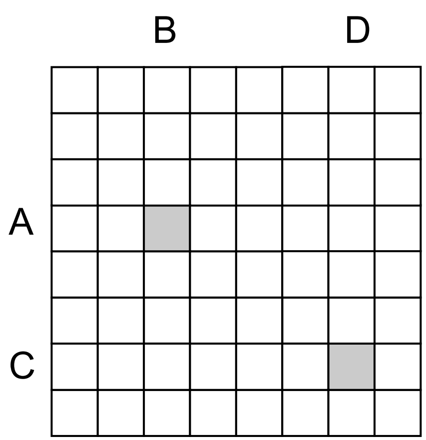


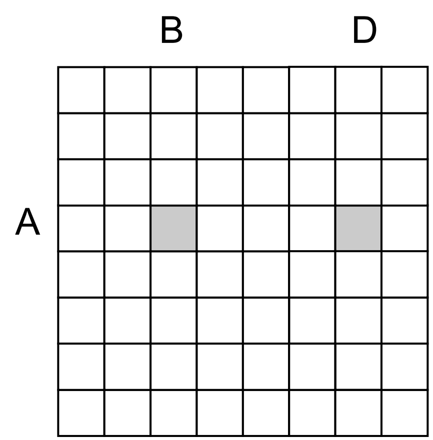
$$\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

- m x n board → two restricted positions??

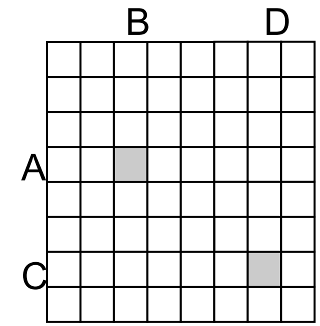
# <u>Taboo Squares</u>

- How to account for <u>two</u> restricted positions

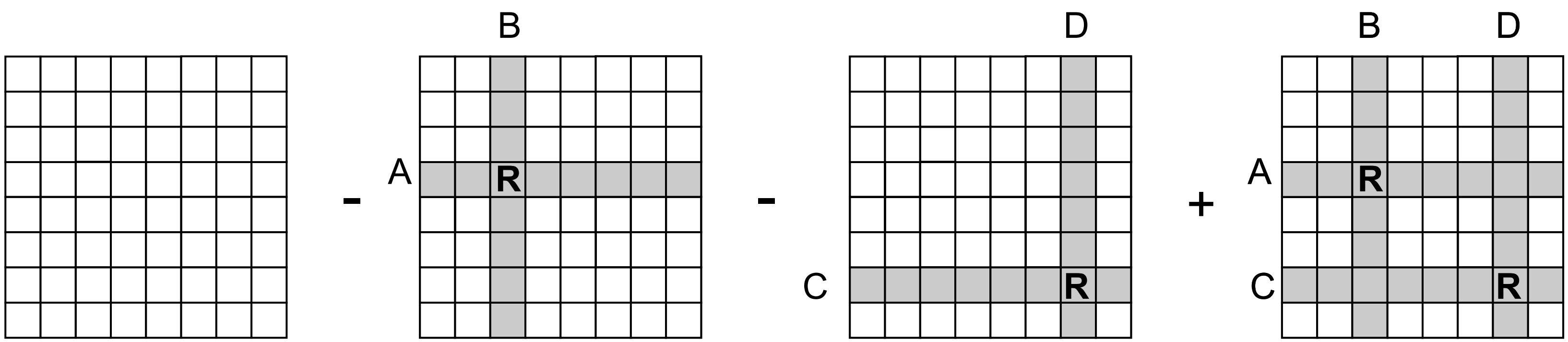


# <u>Taboo Squares</u>




- How to account for <u>two</u> restricted positions
- Idea: "Total – Bad," but there's more to it

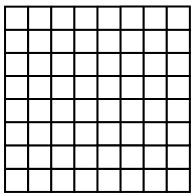



$$\binom{m}{r}\binom{n}{r}r! \quad - \quad \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! \quad - \quad \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! \quad + \quad \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

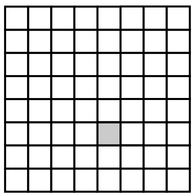
# Taboo Squares

- m x n board → no restricted positions



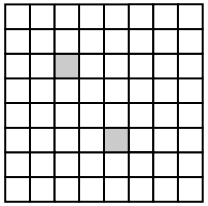
$$\binom{m}{r}\binom{n}{r}r!$$

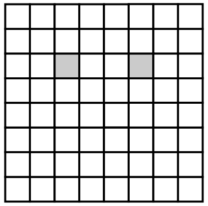
- m x n board → one restricted position



$$\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

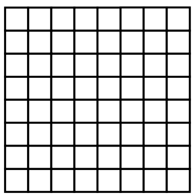
- m x n board → two restricted positions





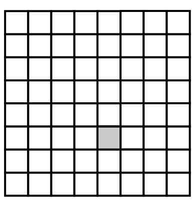
# <u>Taboo Squares</u>

- m x n board → no restricted positions



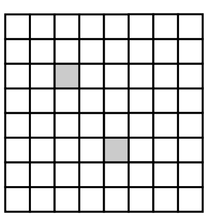
$$\binom{m}{r}\binom{n}{r}r!$$

- m x n board → <u>one</u> restricted position

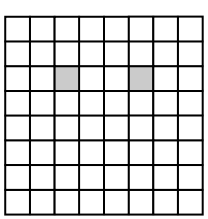


$$\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

- m x n board → <u>two</u> restricted positions



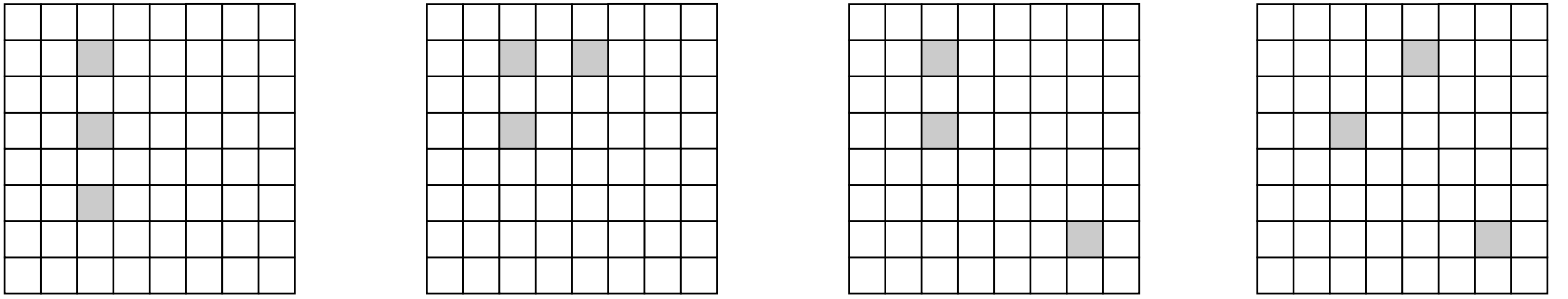
$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$



$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

# Taboo Squares

- Three restricted positions??



  - We should be motivated to find other counting techniques

# Taboo Squares

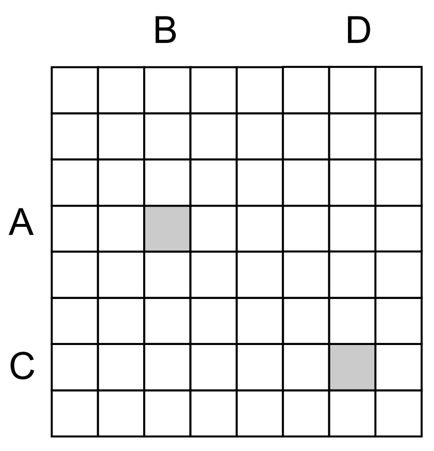
- Inclusion/Exclusion Principle
- This kind of problem serves the purpose of
  - Introducing restricted position
  - Motivating other counting methods

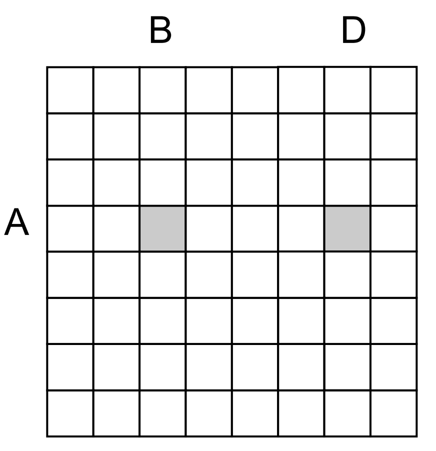
# Now it's your turn

- Part A has been filled out for you
- Use Part A to complete Part B, discussing the activity with someone next to you

# Now it's your turn!

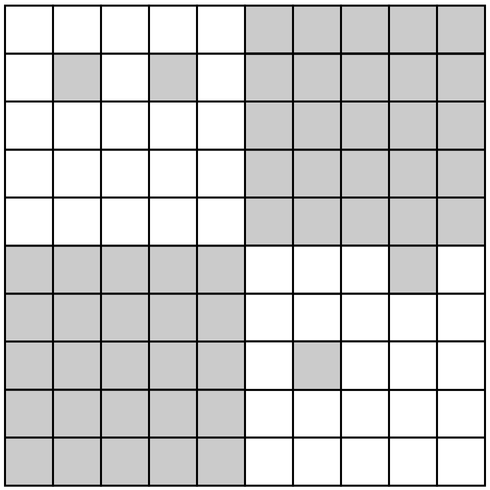
- You've been given the formulas for counting the following two boards.



# Now it's your turn

- What did you discover?



# Rook Kung-Fu

- We want to establish a set of rules that will allow us to count ANY rook board.
- Amazingly, there are just three – count 'em, **THREE** – rules that allow us to do so.
- We call these the "Rook Rules."

# Rook Kung-Fu

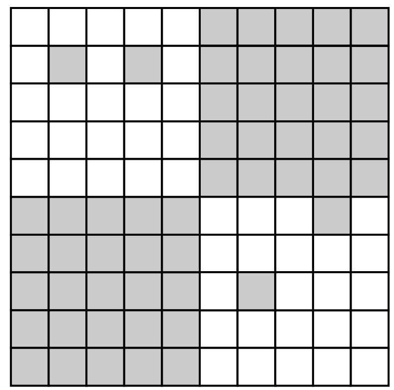
- A little notation

$$n_r(B)$$

= # of ways of placing *r* non-attacking rooks on a board *B*

# Rook Kung-Fu

- Rook Rule #1: Disjoint boards

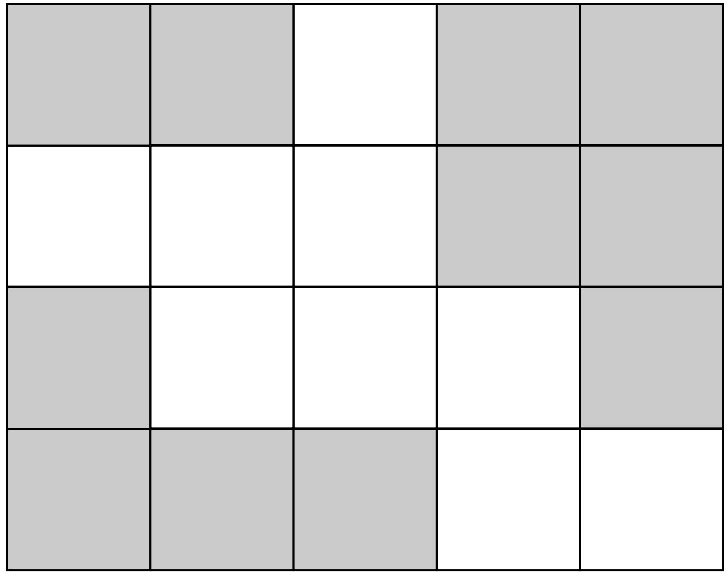


Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap any rows or columns, then

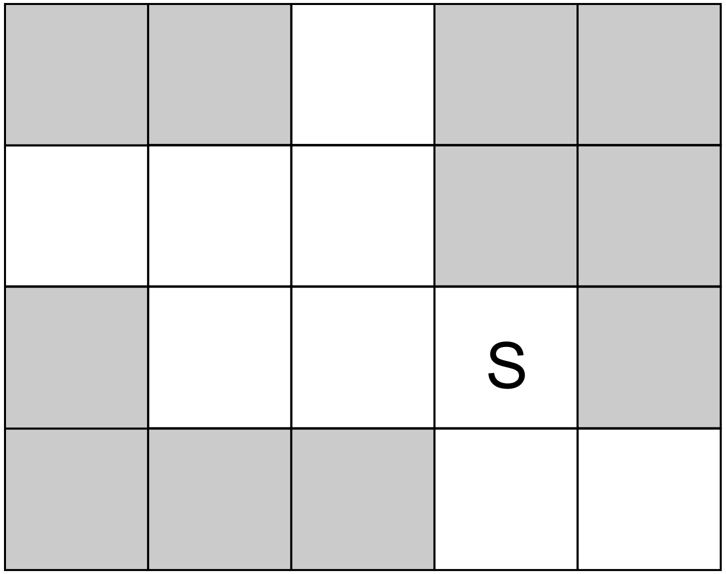
$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

# Rook Kung-Fu

- Rook Rule #2: Use/Don't Use
- Here's how it works…

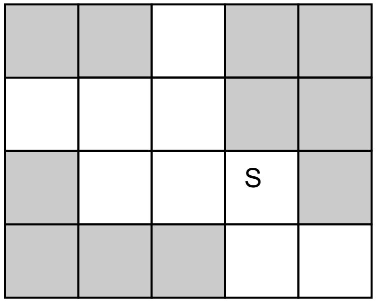


Find number of ways to place $r$ rooks

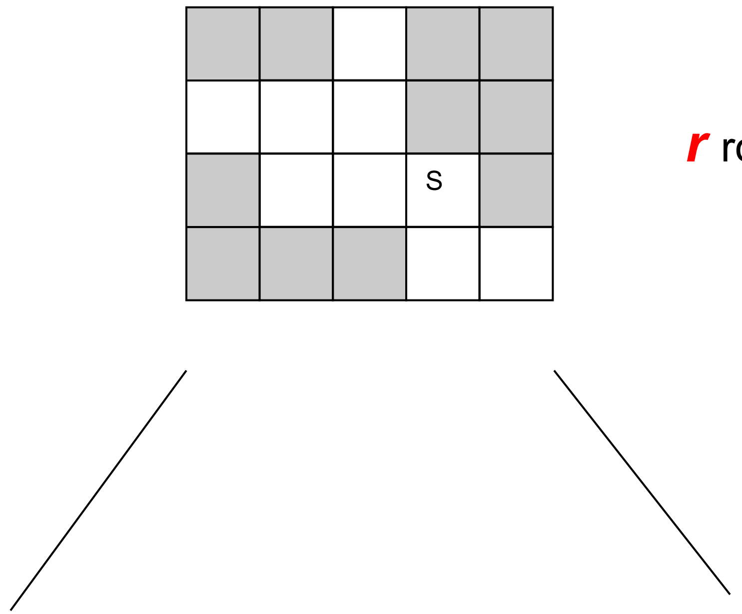


*r* rooks

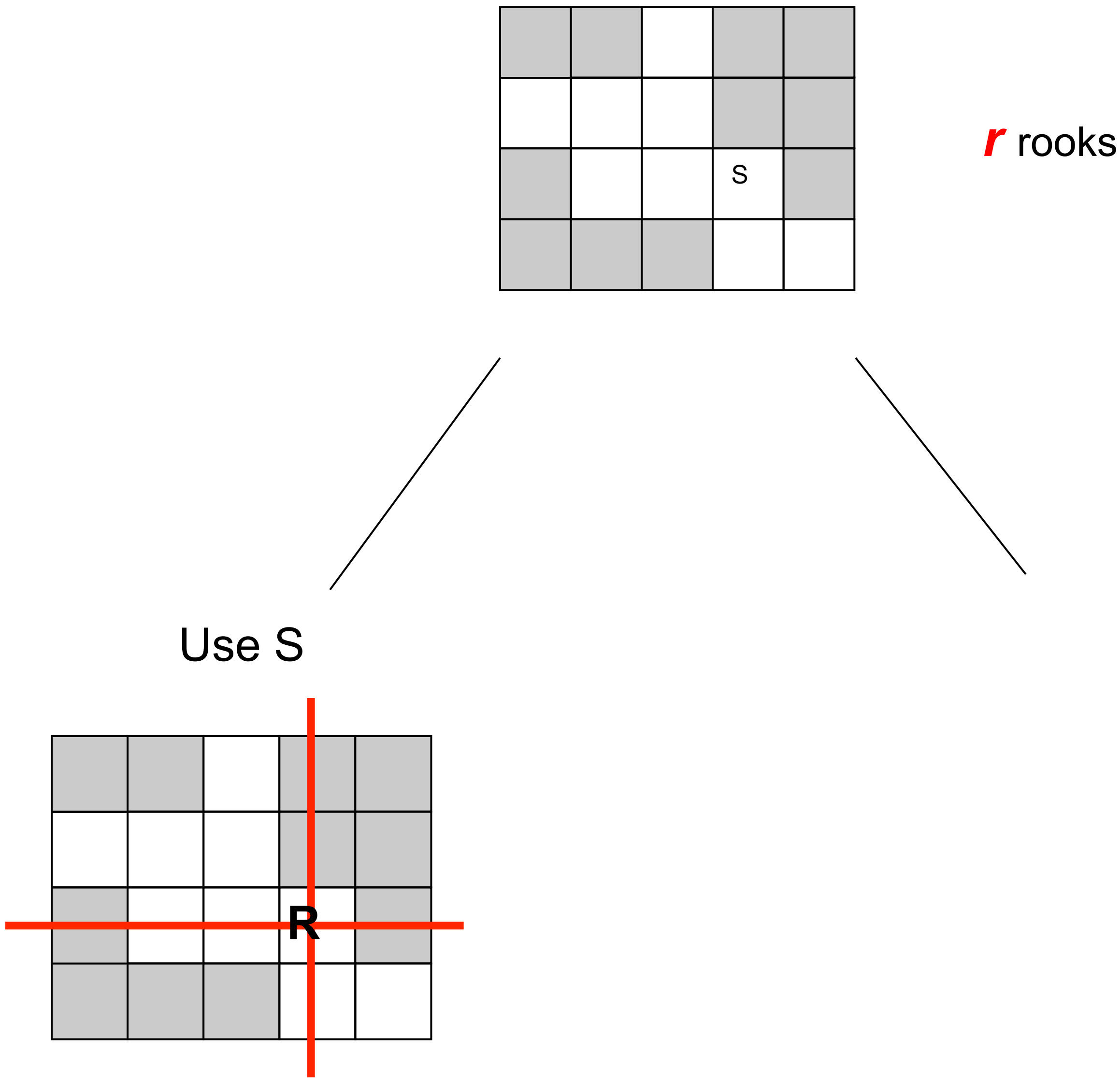
Pick any allowable square S

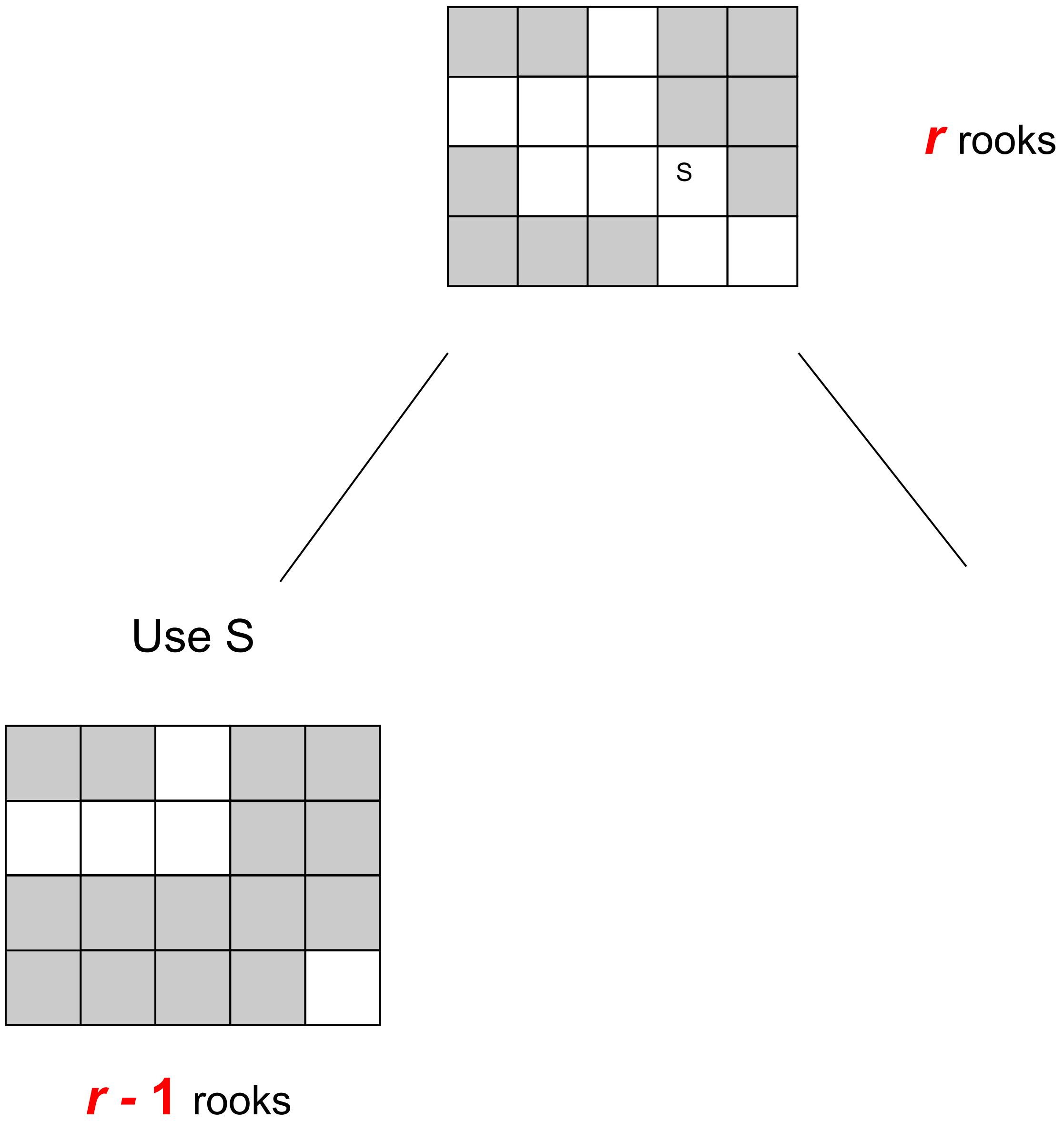


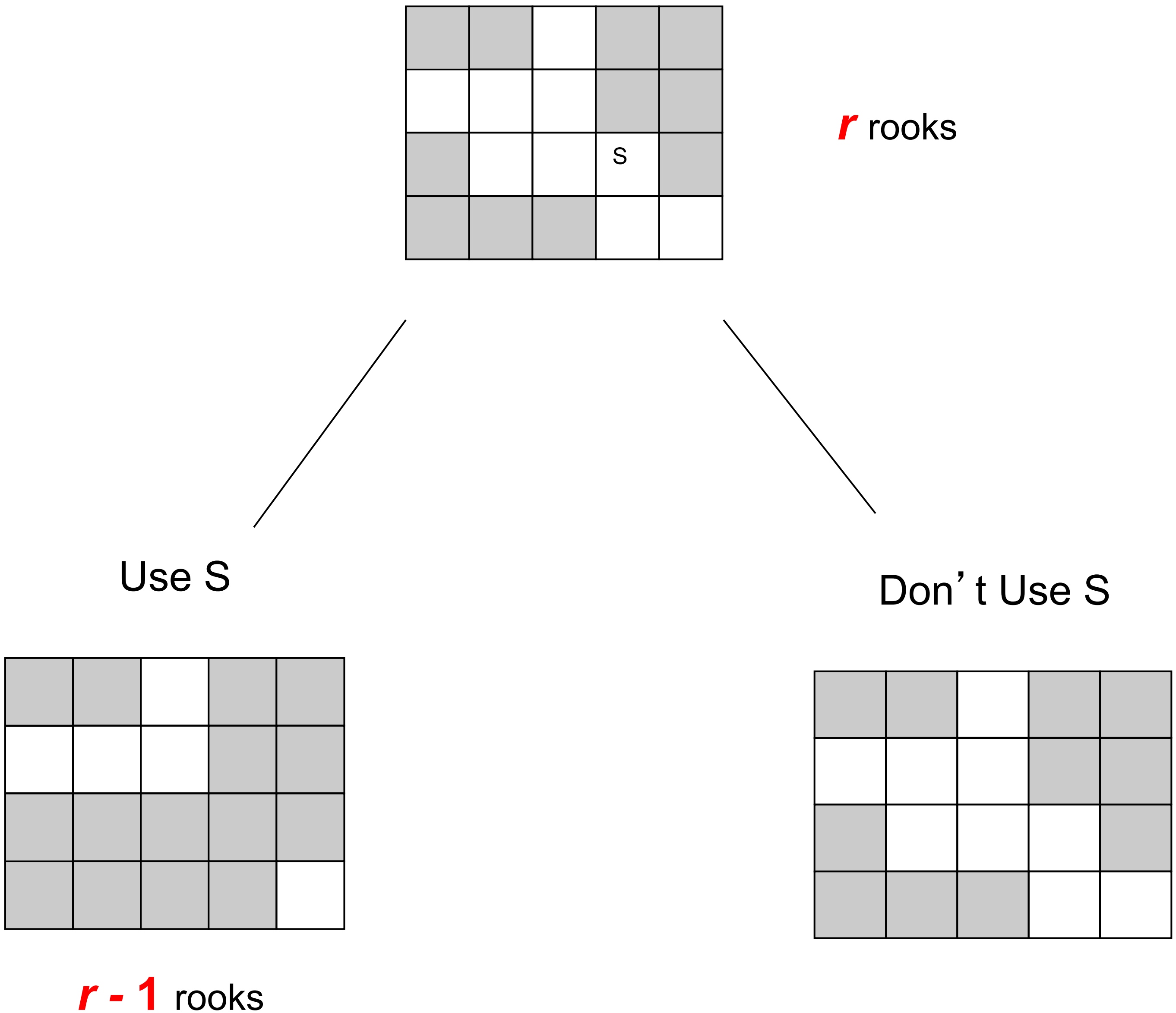

*r* rooks




*r* rooks


S
r rooks
Use S
R


S
r rooks
Use S
r - 1 rooks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r rooks
Use S
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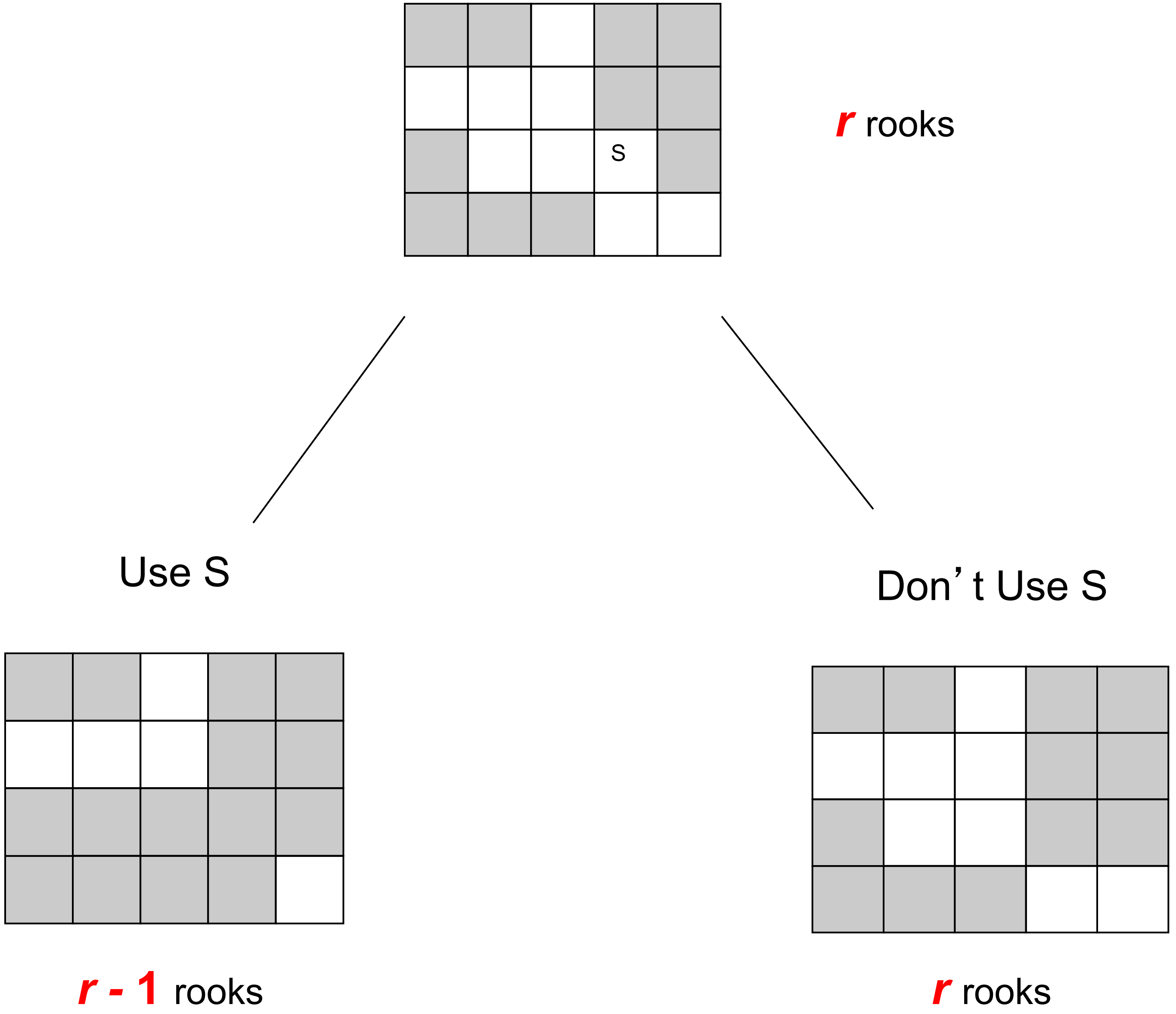
r - 1 rooks


S
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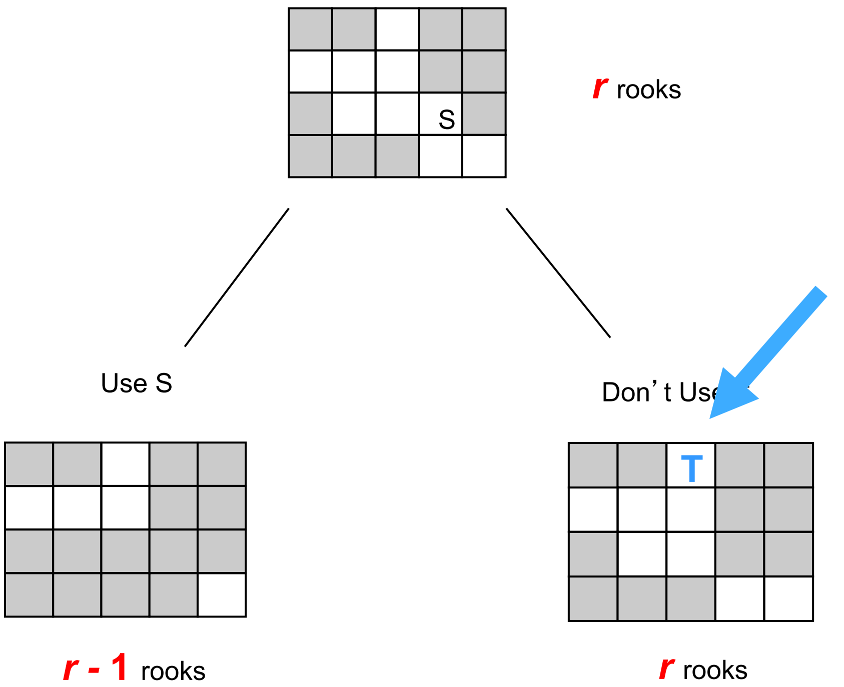
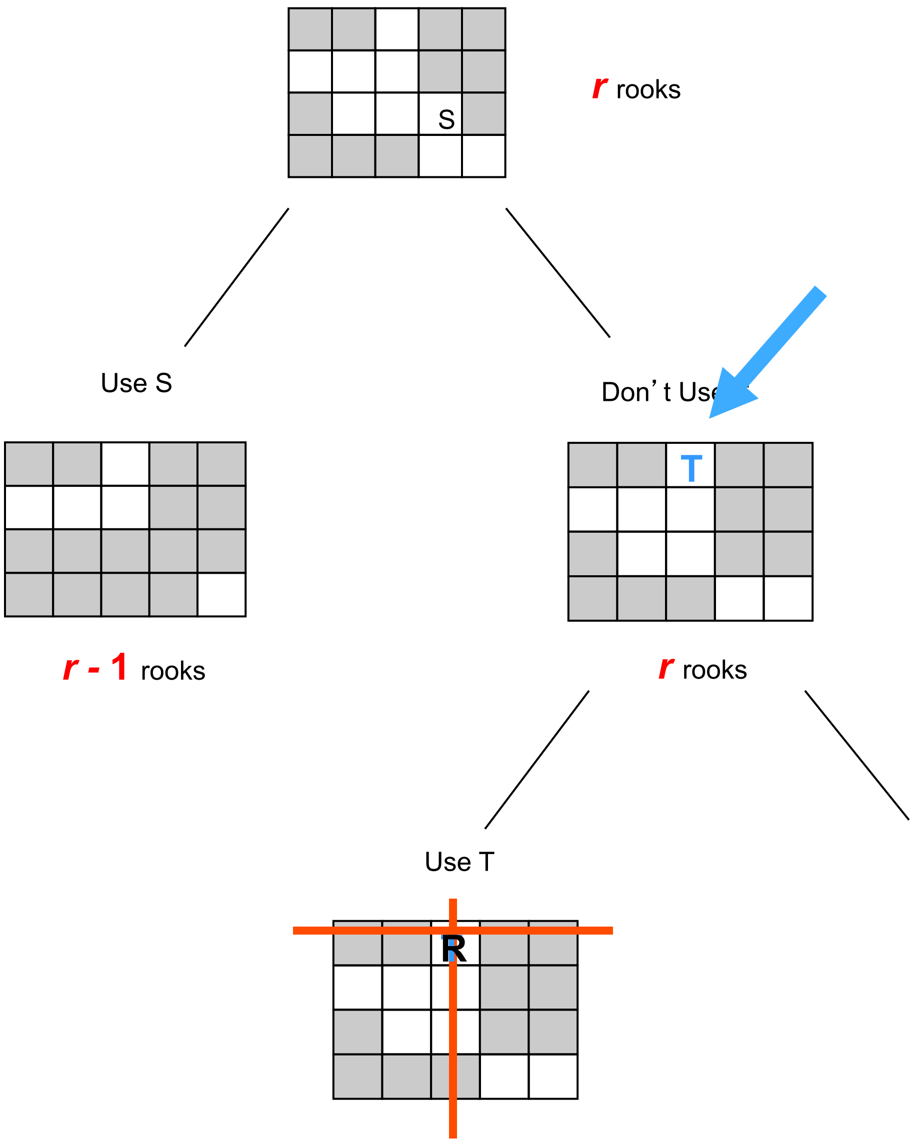
r - 1 rooks
r rooks




Pick a new square T



r rooks
S
Use S
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Use T
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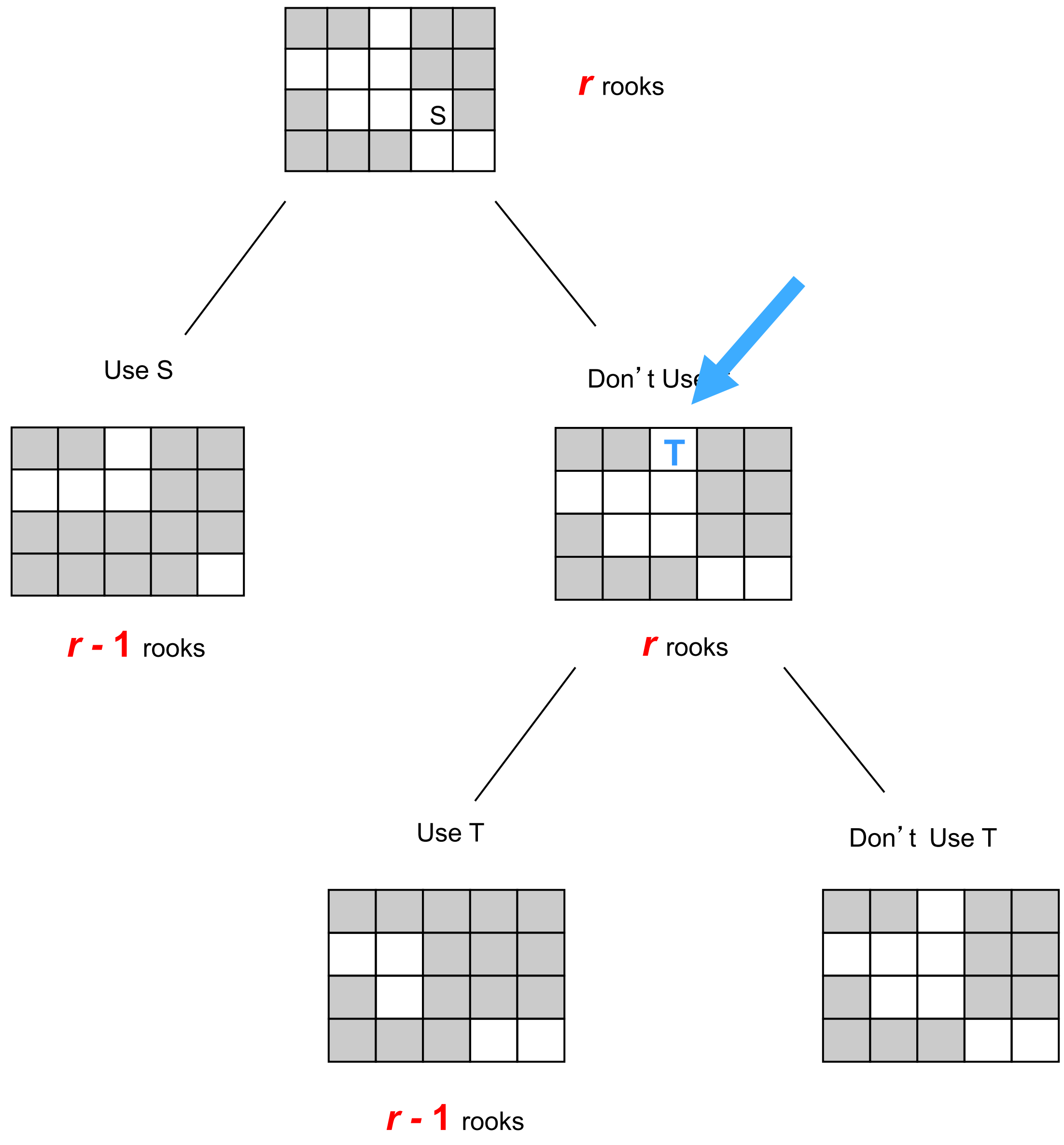

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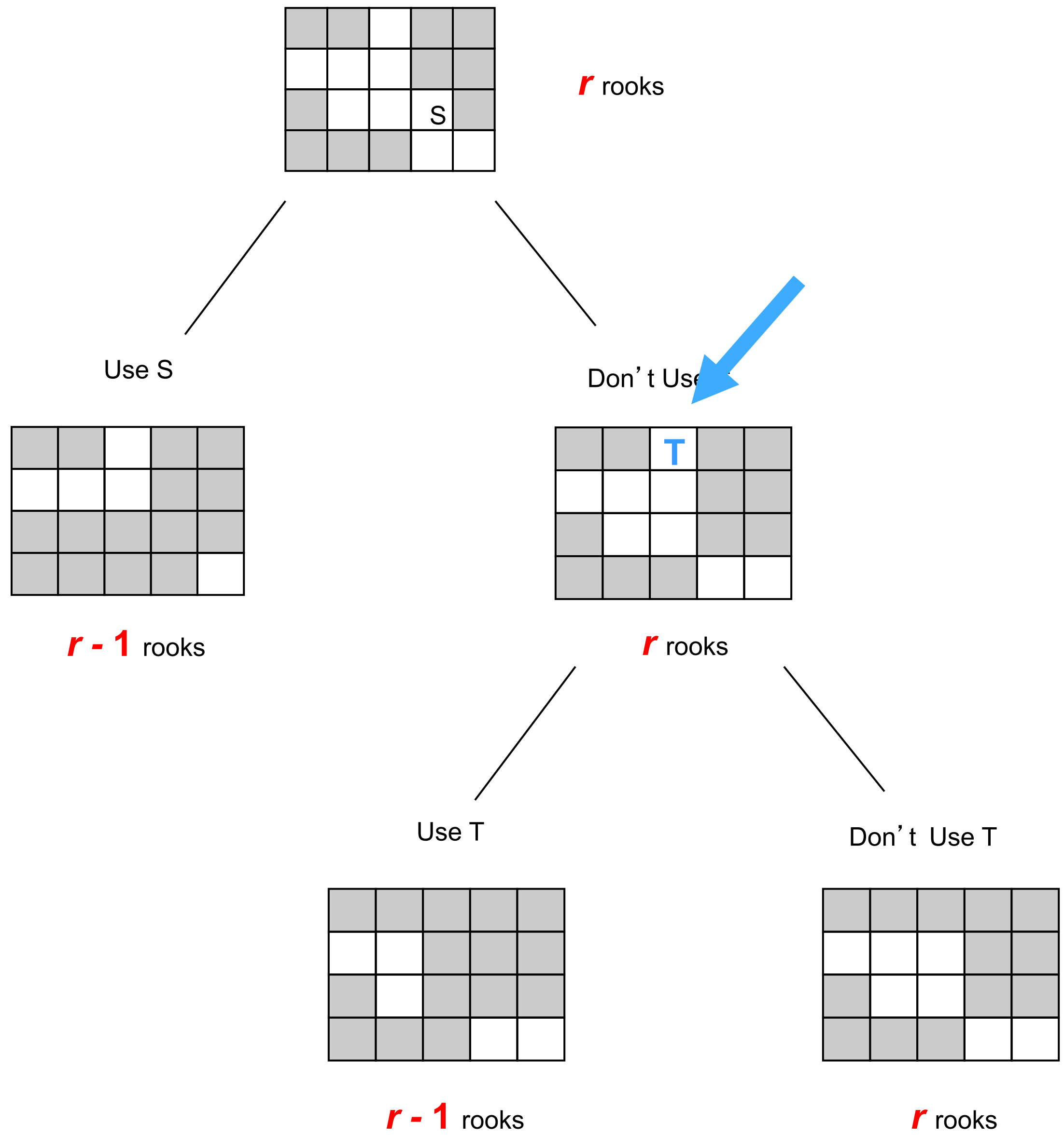
r - 1 rooks



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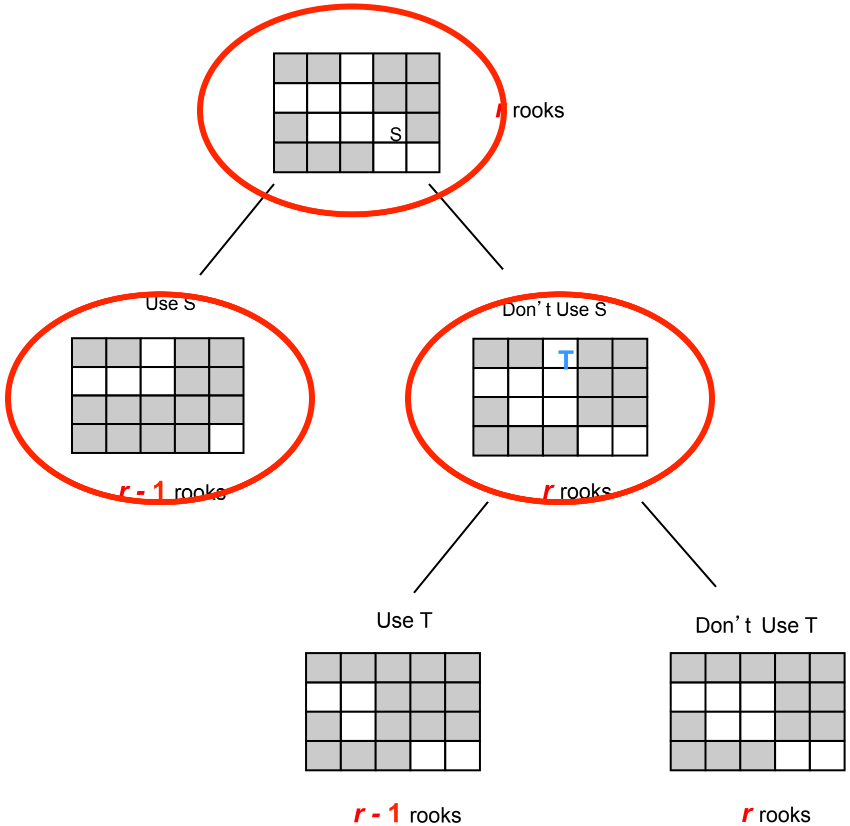
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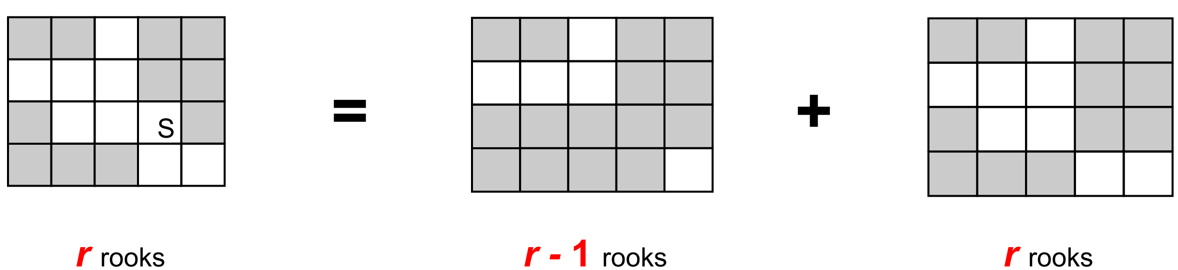
S
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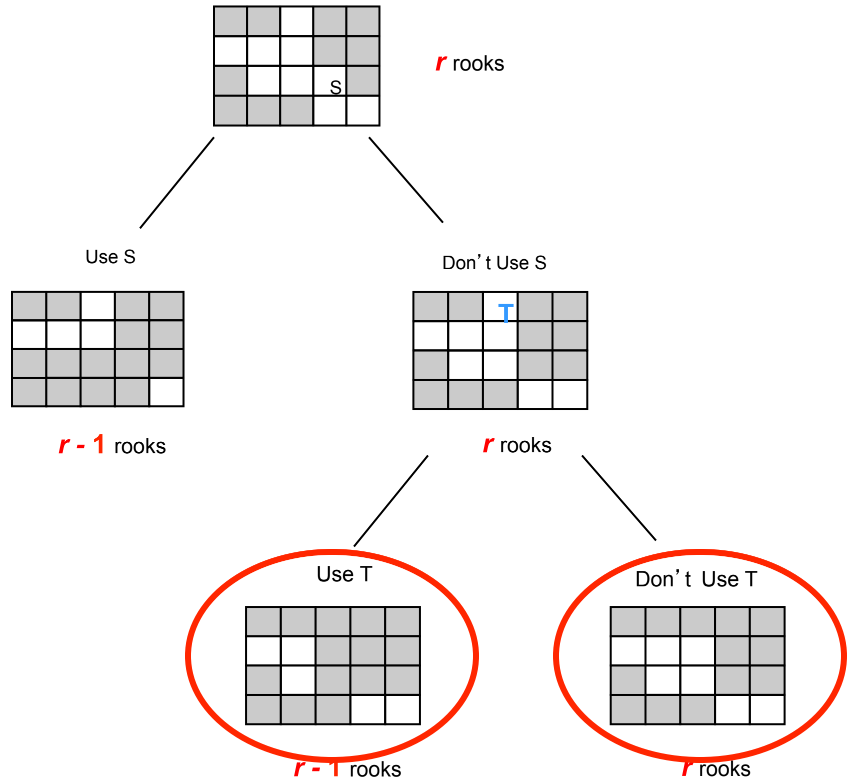

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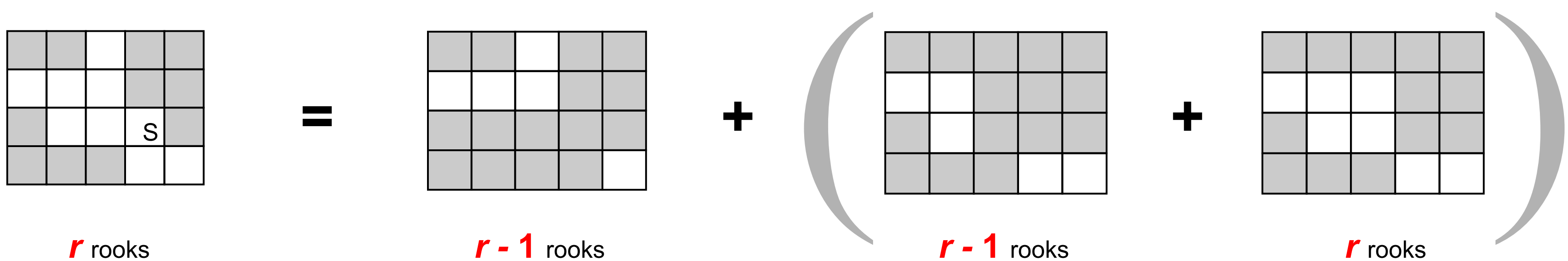

S
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r - 1 rooks
r rooks

# Rook Kung-Fu

- Rook Rule #2: Use/Don't Use

Rook Rule #2: (**Use / Don't Use**) If a square S of a board C is not a forbidden square, then

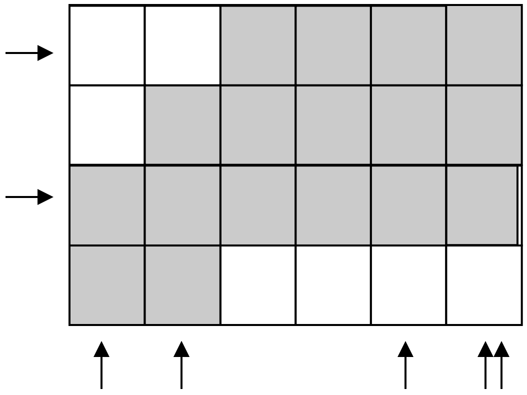
$$n_r(C) = n_{r-1}(C_S) + n_r(C_{not-S}),$$

where $C_S$ is the board formed when we *use* S and $C_{not-S}$ is the board formed when we *don't use* S.

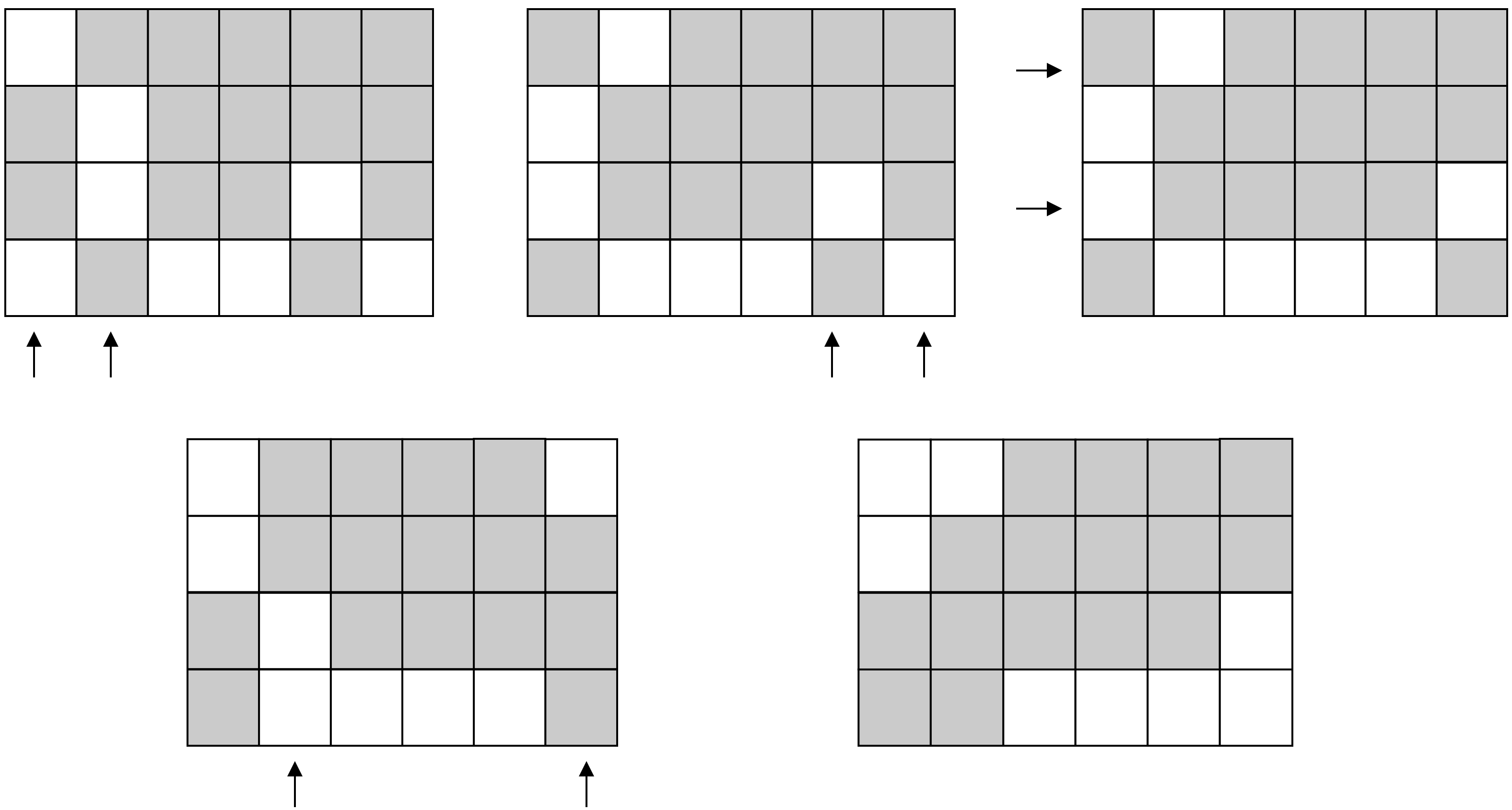
# Rook Kung-Fu

- Rook Rule #3: Switcheroo

# Rook Kung-Fu



# Rook Kung-Fu



# Rook Kung-Fu

- Rook Rule #3: Switcheroo

Rook Rule #3: (**Switcheroo**) Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then for any integer *r*,

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

# Rook Kung-Fu

- So, we have three “Rook Rules” that allow us to count any rook board, regardless of size or number of restricted positions!
  - Rook Rule #1 – Disjoint Boards
  - Rook Rule #2 – Use/Don’t Use
  - Rook Rule #3 – Switcheroo

# Rook Polynomials

- We' re now ready for the official definition of Rook Polynomials
  - Recall that the number of ways of placing *r* rooks on a board *B* is denoted by

$$n_r(B)$$

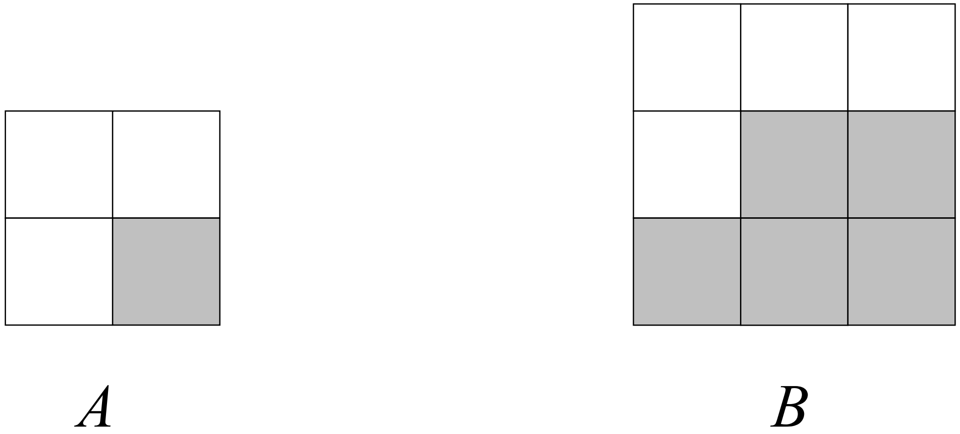
  - Then we define the **rook polynomial** of a board *B* to be the polynomial

$$R(B,x) = \sum_{r \geq 0} n_r(B) x^r$$

# Rook Polynomial:

$$R(B,x) = \sum_{r \geq 0} n_r(B) x^r$$

- Let's compute the Rook Polynomials of *A* and *B* below



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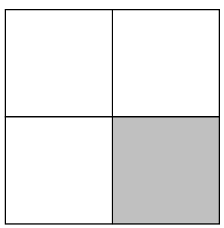
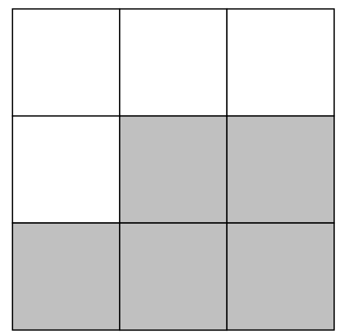
$$R(B,x) = \sum_{r \geq 0} n_r(B) x^r$$

- Let's compute the Rook Polynomials of *A* and *B* below



*A*



*B*

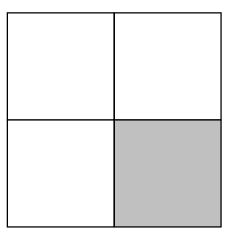
$R(A,x) = 1x^0 + 3x^1 + 1x^2$

$R(B,x) = 1x^0 + 4x^1 + 2x^2 + 0x^3$

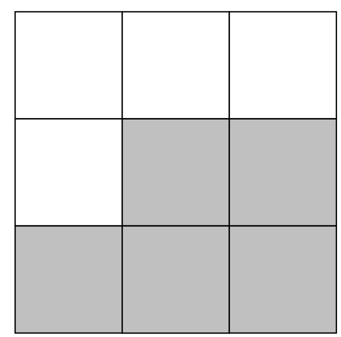
# Rook Polynomial:

$$R(B,x)=\sum_{r\geq 0} n_r(B)x^r$$

- Let's compute the Rook Polynomials of *A* and *B* below



*A*



*B*

$R(A,x) = 1 + 3x + x^2$

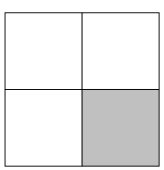
$R(B,x) = 1 + 4x + 2x^2$

# Rook Polynomials

- Each of the Rook Rules has a "Rook Polynomial" version
  - Rook Rule #1 – Disjoint Boards
  - Rook Rule #2 – Use/Don't Use
  - Rook Rule #3 – Switcheroo

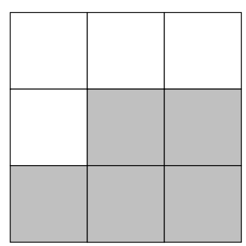
# Rook Polynomials

- Rook Rule #1 – Disjoint Boards
  - Take a moment to multiply these polynomials together



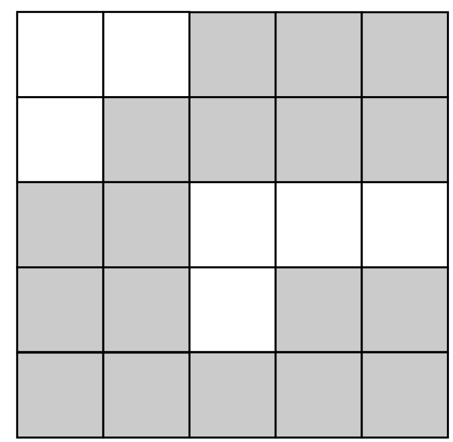
*A*

$R(A,x) = 1 + 3x + x^2$



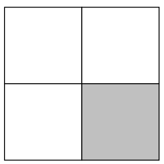
*B*

$R(B,x) = 1 + 4x + 2x^2$



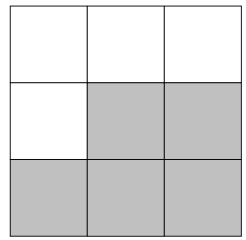
C

# Rook Polynomials



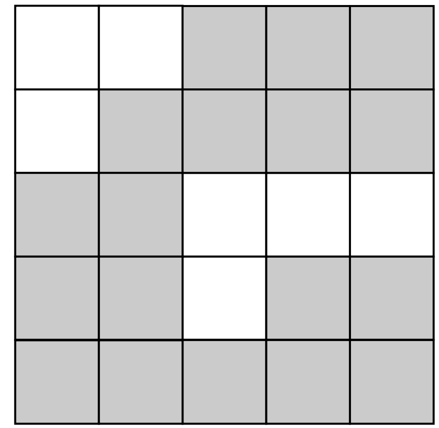
*A*

R(A,x) = $1 + 3x + x^2$



*B*

R(B,x) = $1 + 4x + 2x^2$



C

R(C,x) $= (1 + 3x + x^2)(1 + 4x + 2x^2)$

$= 1 + 4x + 2x^2 + 3x + 12x^2 + 6x^3 + x^2 + 4x^3 + 2x^4$

$= (1 \cdot 1) + (1 \cdot 4 + 3 \cdot 1)x + (1 \cdot 2 + 3 \cdot 4 + 1 \cdot 1)x^2 + (3 \cdot 2 + 1 \cdot 4)x^3 + (1 \cdot 2)x^4$

$= 1 + 7x + 15x^2 + 10x^3 + 2x^4$

# Rook Polynomials

Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap any rows or columns, then

$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

---

Rook Rule #1**: (polynomial version)** If a board C consists of two sub-boards A and B that do not overlap any rows or columns, then

$$R(C,x) = R(A,x)R(B,x)$$

# Rook Polynomials

Rook Rule #2: (**Use / Don't Use**) If the i,j-square S of a board C is not a forbidden square, then

$$n_r(C) = n_{r-1}(C_S) + n_r(C_{not-S}),$$

where $C_S$ is the board formed when we *use* S, and $C_{not-S}$ is the board formed when we *don't use* S.

---

Rook Rule #2: (**polynomial version**) If the i,j-square S of a board C is not a forbidden square, then

$$R(C,x) = xR(C_S,x) + R(C_{not-S},x),$$

where $C_S$ is the board formed when we *use* S, and $C_{not-S}$ is the board formed when we *don't use* S.

# Rook Polynomials

Rook Rule #3: (**Switcheroo**) Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

---

Rook Rule #3**: (polynomial version)** Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$R(B, x) = R(C, x)$$

In other words, we can swap rows and columns without affecting the outcome.

# Rook Polynomials
# Some Practice…

- Five kids want the last five pets at a pet store:
  a hamster, a frog, a goldfish, a cockatiel, and a puppy.

  - Carly only wants something with fur (feathers don't count).
  - Sarah prefers amphibians.
  - Brad would like anything that doesn't have claws or talons.
  - Joanna only wants a puppy or a hamster.
  - Derek wants a pet that can fly.

- Set up a rook board for this problem and find its rook polynomial.

# Rook Polynomials

- Carly only wants something with fur (feathers don’t count)
- Sarah prefers amphibians.
- Brad would like anything that doesn’t have claws or talons.
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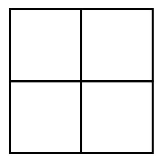
| | H | F | G | C | P |
|---|---|---|---|---|---|
| C | | | | | |
| S | | | | | |
| B | | | | | |
| J | | | | | |
| D | | | | | |

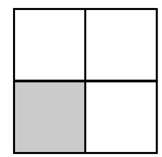
| | H | P | G | F | C |
|---|---|---|---|---|---|
| C | | | | | |
| J | | | | | |
| B | | | | | |
| S | | | | | |
| D | | | | | |

# Rook Polynomials

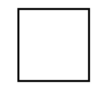
- Three disjoint boards, whose rook polynomials are easy to compute.



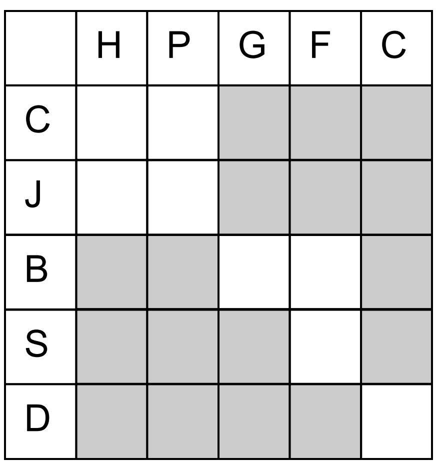
$1 + 4x + 2x^2$



$1 + 3x + x^2$



$1 + x$




$$= (1 + 4x + 2x^2)(1 + 3x + x^2)(1 + x)$$

$$= 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$$

# You *thought* you were getting practice at finding rook polynomials

but really…

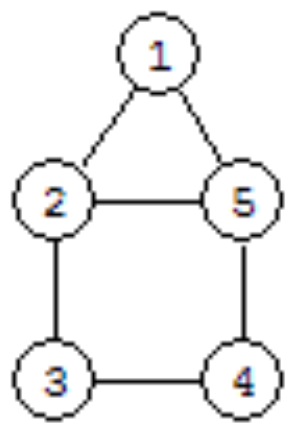
You were in the wonderful world of **generating functions**

# Generating Functions

- A **generating function** of a sequence is a polynomial (or power series) whose coefficients are the terms of the sequence.
- Important operations on sequences often correspond to simple algebraic manipulations of the generating functions.
- Rooks provide a natural context in which to present these.

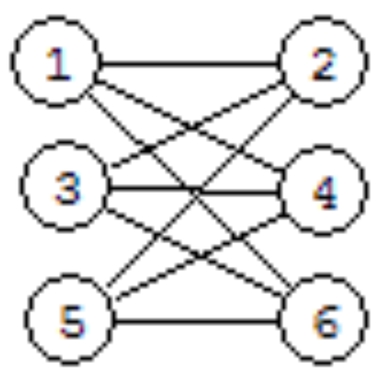
# More Mathematical Connections

- A **graph** is a set of edges and a set of vertices with some incidence relation.
- Example



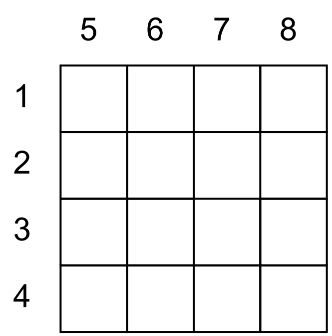
# More Mathematical Connections

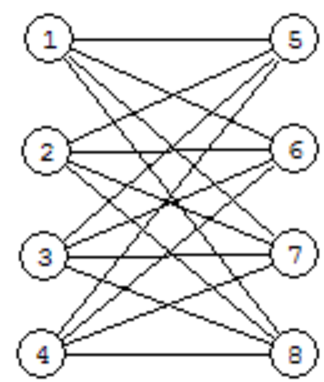
- A **graph** is a set of edges and a set of vertices with some incidence relation.
- We say a graph is **bipartite** if its vertices can be separated into two cells, where the vertices of each cell are mutually non-adjacent.
- Example



# More Mathematical Connections

- For a given rook board, we can draw a graph associated with it



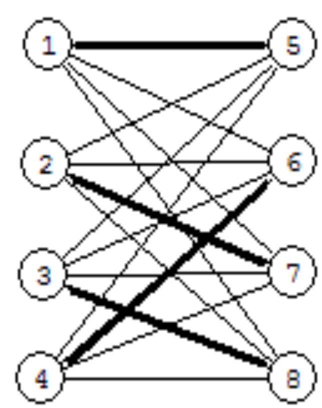




  - The two cells of vertices represent rows and columns, respectively
  - Each edge represents an allowable square

# Matchings

- A **matching** is a set of edges where no two edges share a common vertex.
- A matching corresponds to a non-attacking configuration of rooks
- A matching of size *r* is called an *r* - matching

| | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| 1 | R | | | |
| 2 | | | R | |
| 3 | | | | R |
| 4 | | R | | |



# Matchings

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| 1 | R | | | |
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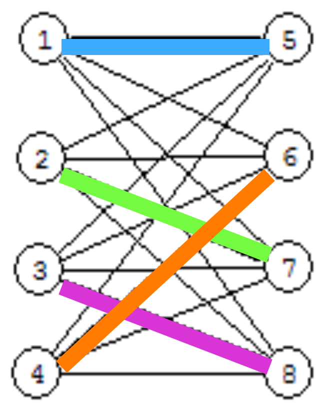
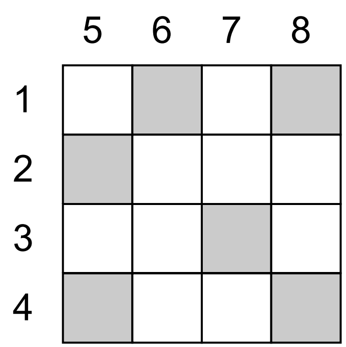
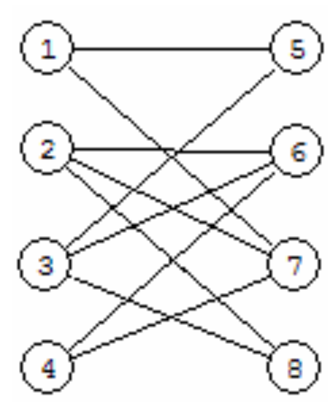
| 3 | | | | R |
| 4 | | R | | |

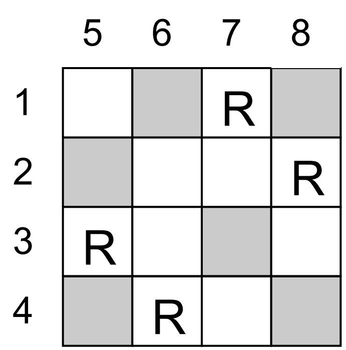


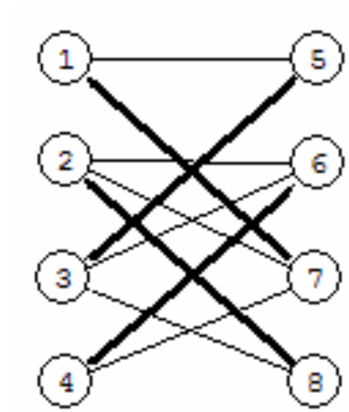
# Matchings

- Note, we can make graphs of boards with restricted positions as well



# Matchings

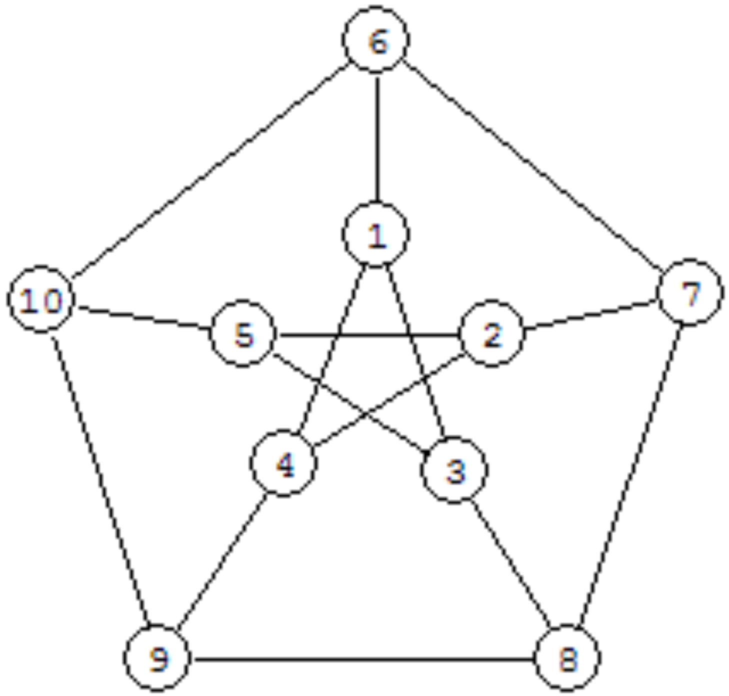
- The number of *r*-matchings in a graph *G* is denoted by

$$m_r(G)$$

- Then the **matchings polynomial** is defined to be

$$\mu(G,x) := \sum_{r\geq 0} m_r(G)x^r$$

# Matchings




$$\mu(G,x) = 1 + 15x + 75x^2 + 145x^3 + 90x^4 + 6x^5$$

# The Rook Trifecta

- Counting Principles
- Generating Functions
- Matchings and Graph Theory

- Combinatorics consists of two major components
    - Enumeration
    - Graph Theory
- Rooks provide a nice marriage of these two ideas

# Rook Topics

- Counting Principles
  - Addition rule/multiplication rule
  - Inclusion/Exclusion
- Generating Functions
- Matchings
- Orthogonal Polynomials
- Recurrences
- Latin Squares
- Permutations
- Derangements

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