Gwen’s Game

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- Tiles must stay within the boundaries of the board.

- Game continues until either (a) all tiles have been played; or (b) no player can legally play a tile

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Example:

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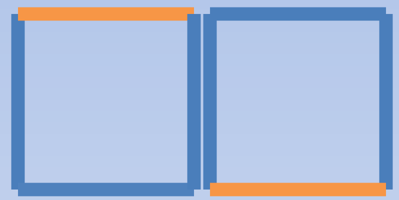
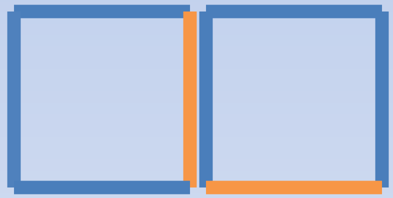


A legal move



Not a legal move

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•Choose two colored pens and create two full sets of tiles. Each player gets a full set of tiles.

•When you’re done, come show me. I’ll give you a game board on which to play.

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You found the 6 following possible tiles:

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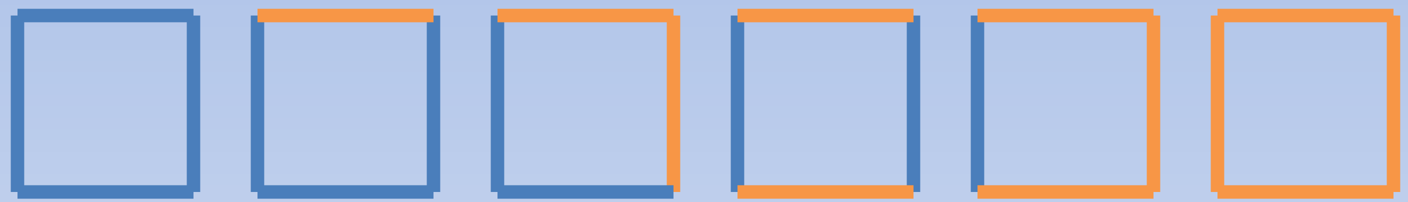

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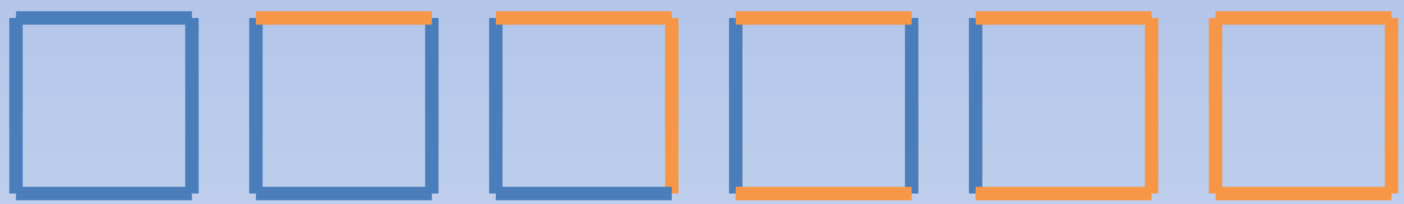
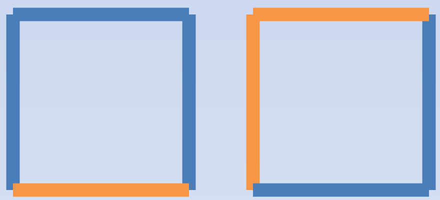


Question: Why don’t we have these two in the set?



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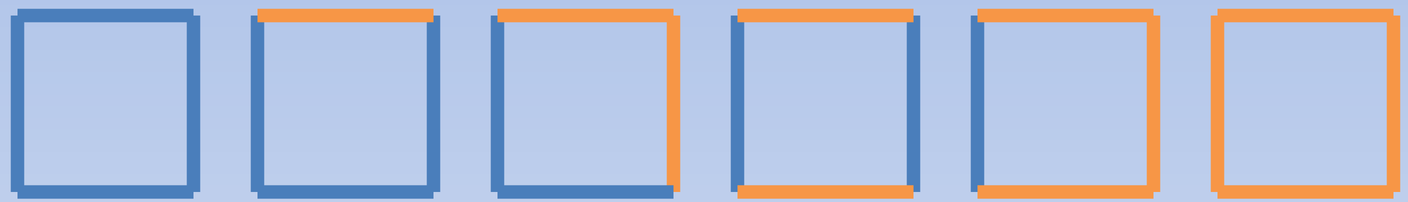

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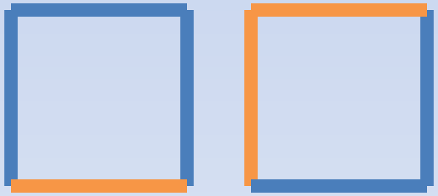

Answer: We already have them. If you can rotate a piece to make a new one, then you already have it in the set.

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Suppose instead that the game tiles cannot be rotated, so that the two tiles below are considered different.

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Now how many possible tiles are there in a set? Try to draw them out.

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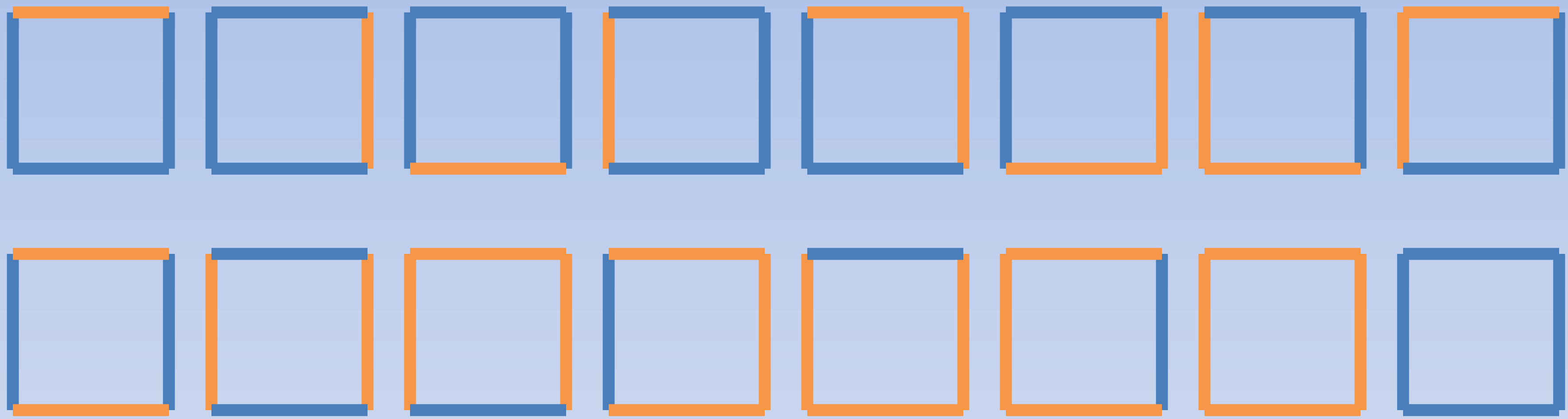
This time you found 16 different tiles.



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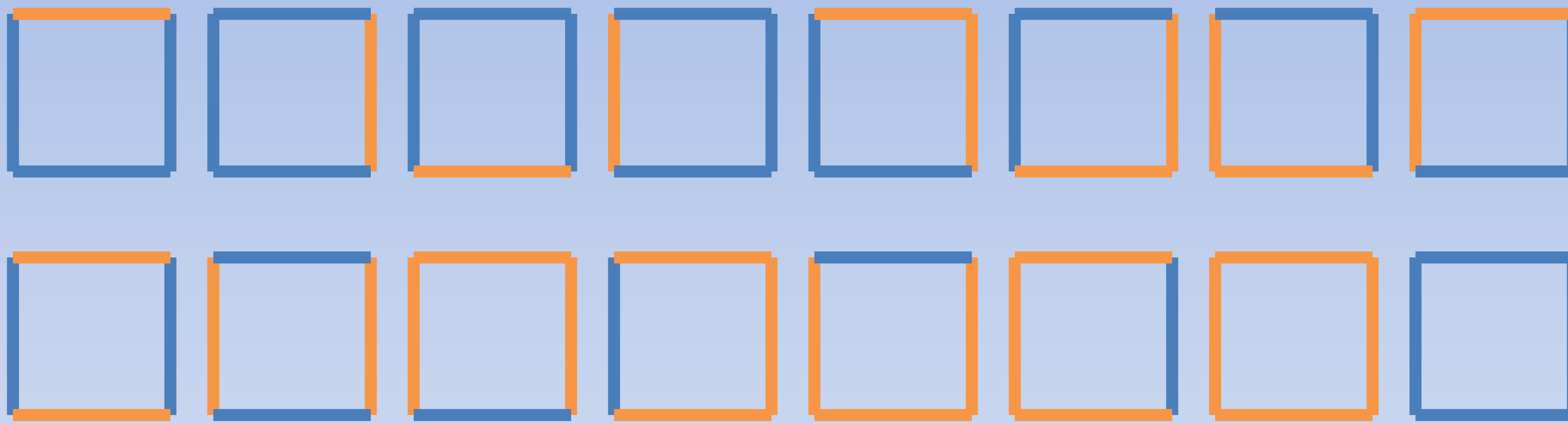

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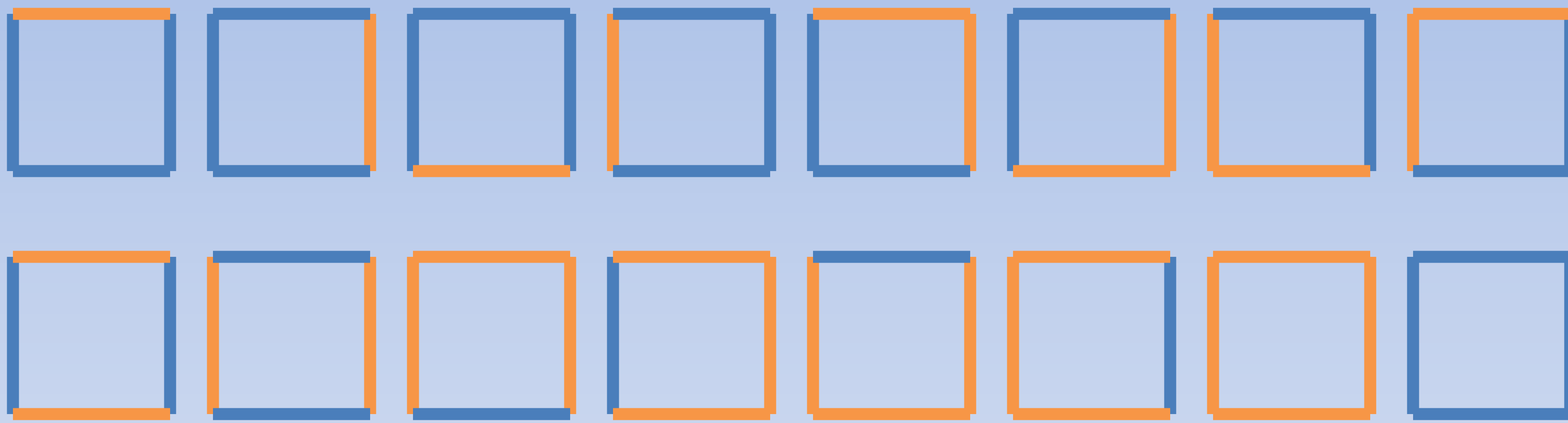

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At each edge, there were 2 possibilities. So going around the shape gave us 2*2*2*2 possible ways to color the edges.

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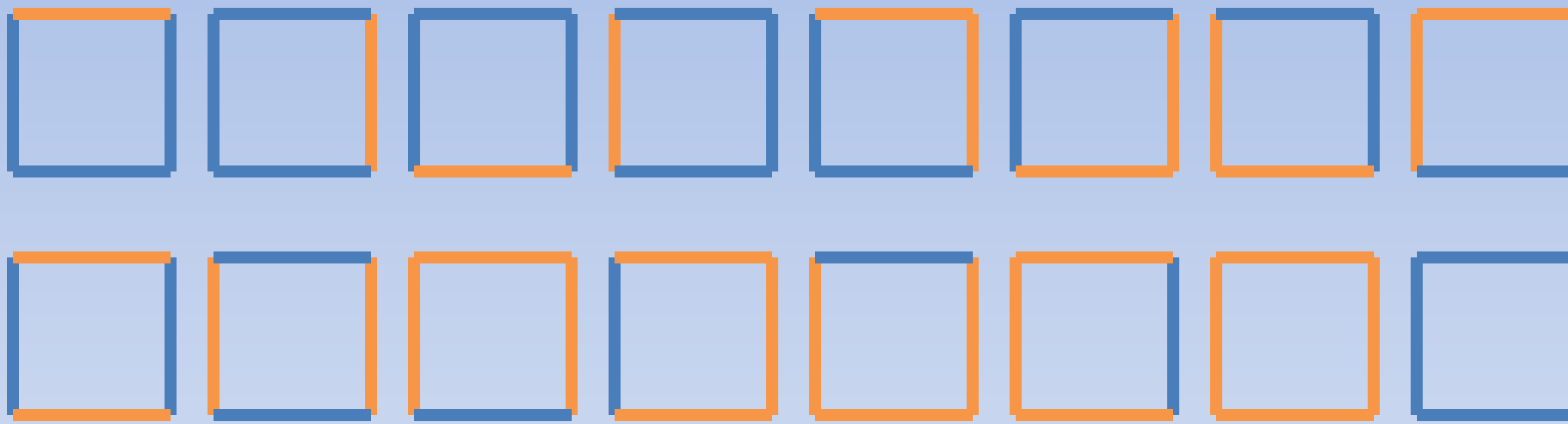

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At each edge, there were 2 possibilities. So going around the shape gave us 2*2*2*2 possible ways to color the edges.

So, is there an easy way to count them when rotations were allowed? What's the idea there?

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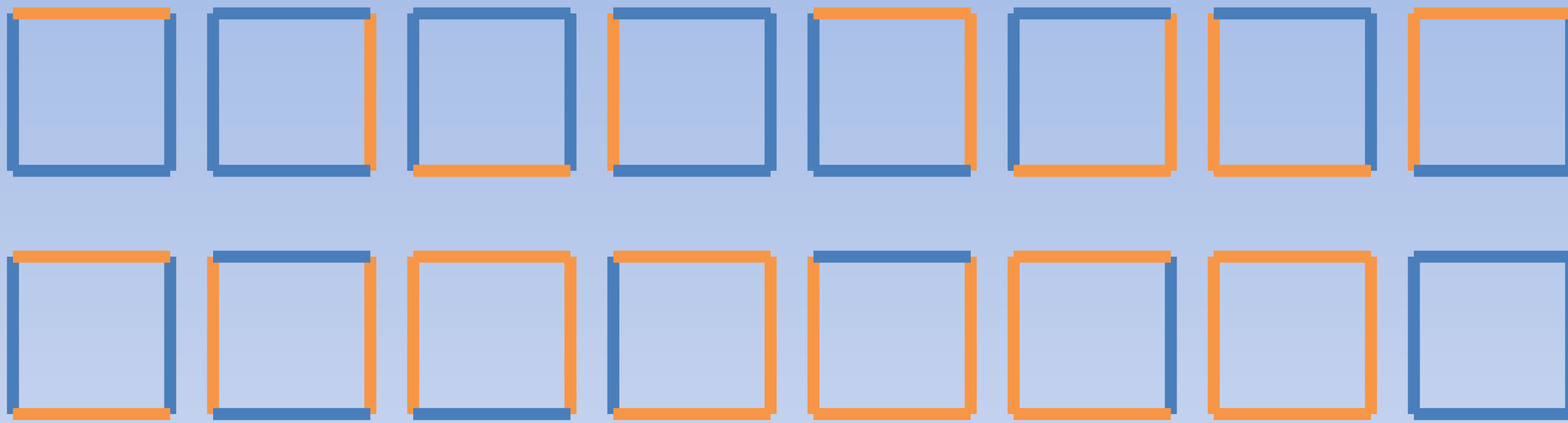

When we counted the original 6 game pieces, we had to decide whether we had two pieces where one could be rotated to make the other. The key here was knowing exactly *how* a square could be rotated: 90^o or 180^o or 270^o

Easy idea, but this is where it all happens.

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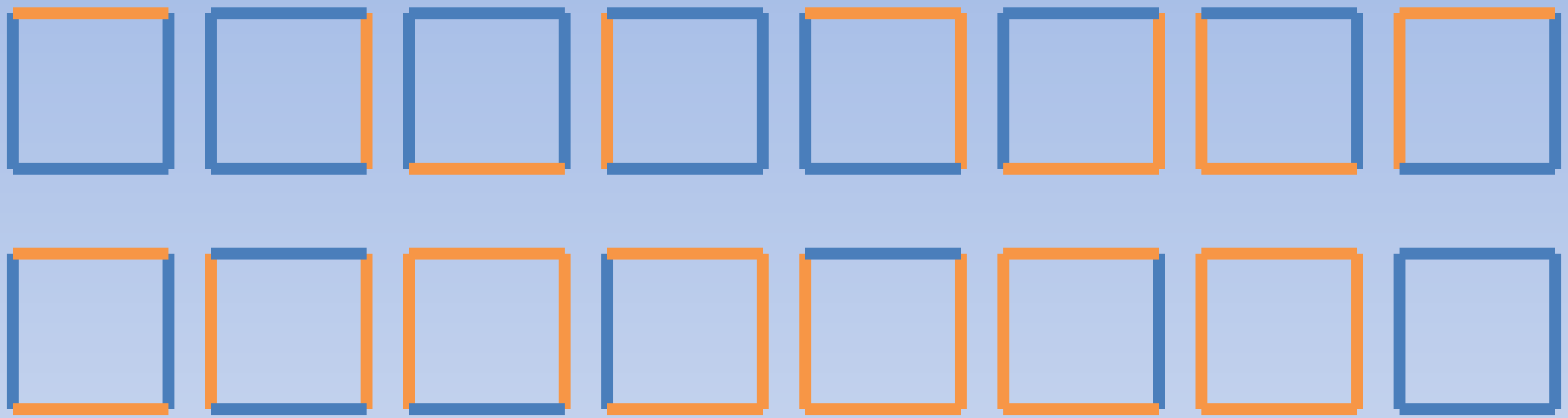

In order to find the 6 colorings which are different, notice that we can think about *partitioning* this set into subsets of tiles which can be rotated into one another.

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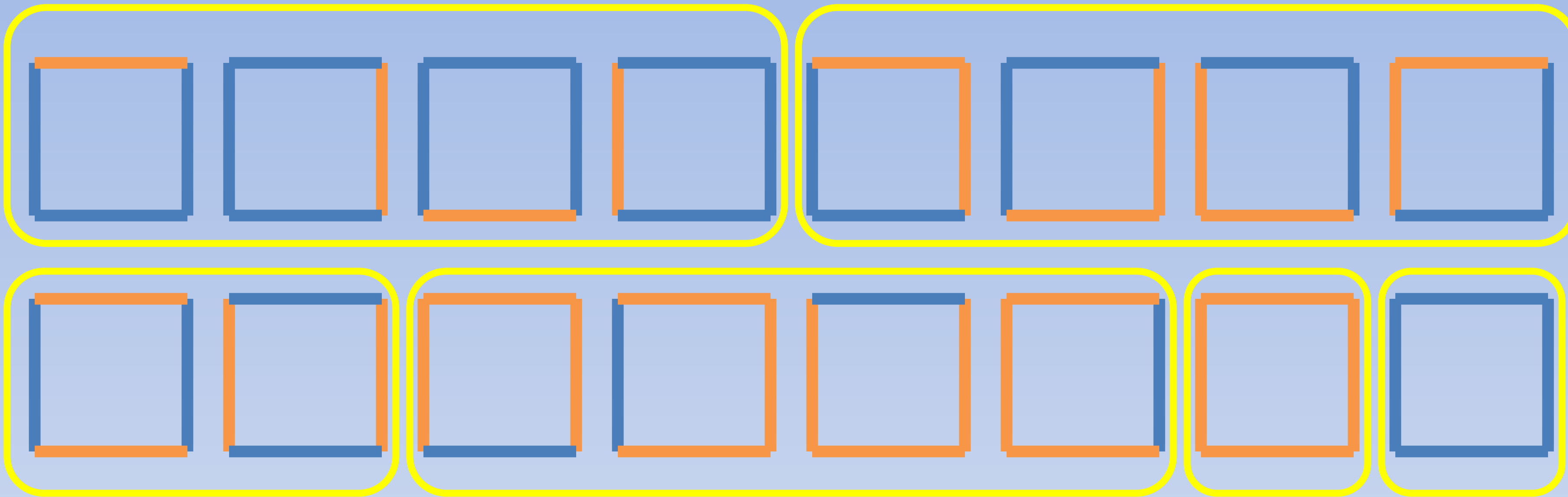

In order to find the 6 colorings which are different, notice that we can think about *partitioning* this set into subsets of tiles which can be rotated into one another.

Now it’s easy to see the 6 distinct tiles – just pick one from each circled subset. Tiles in the same subset are called *equivalent under rotation*.

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So, what if we wanted to play with tiles which are not squares, but regular pentagons (5-sided)?

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   2*2*2*2*2 = 32 possible tiles

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1. Suppose we cannot rotate the tiles. How many possible tiles are there with two possible colors on each edge?

   2*2*2*2*2 = 32 possible tiles

2. Suppose instead that we *can* rotate tiles. Try to draw the 8 distinct tiles.

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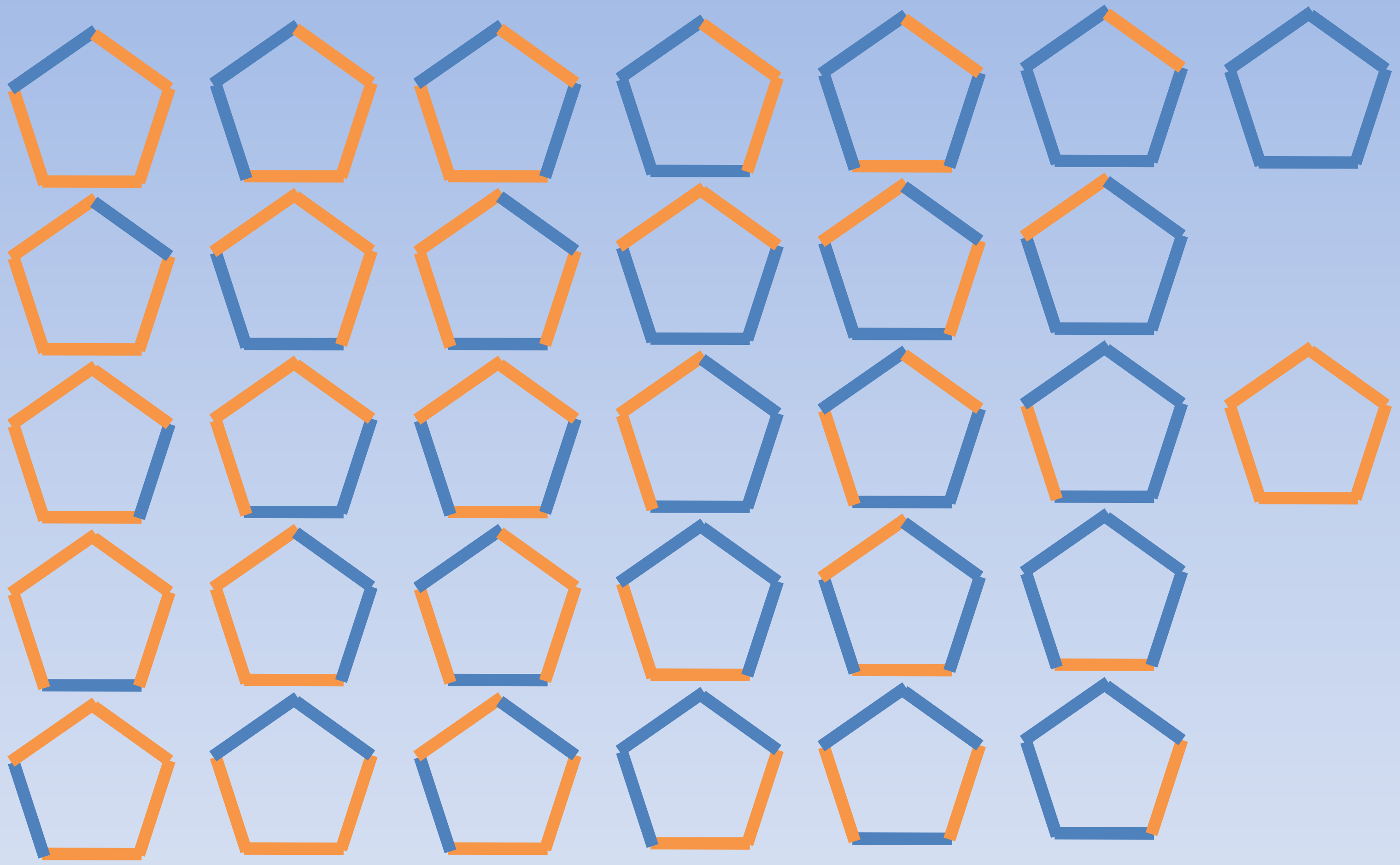

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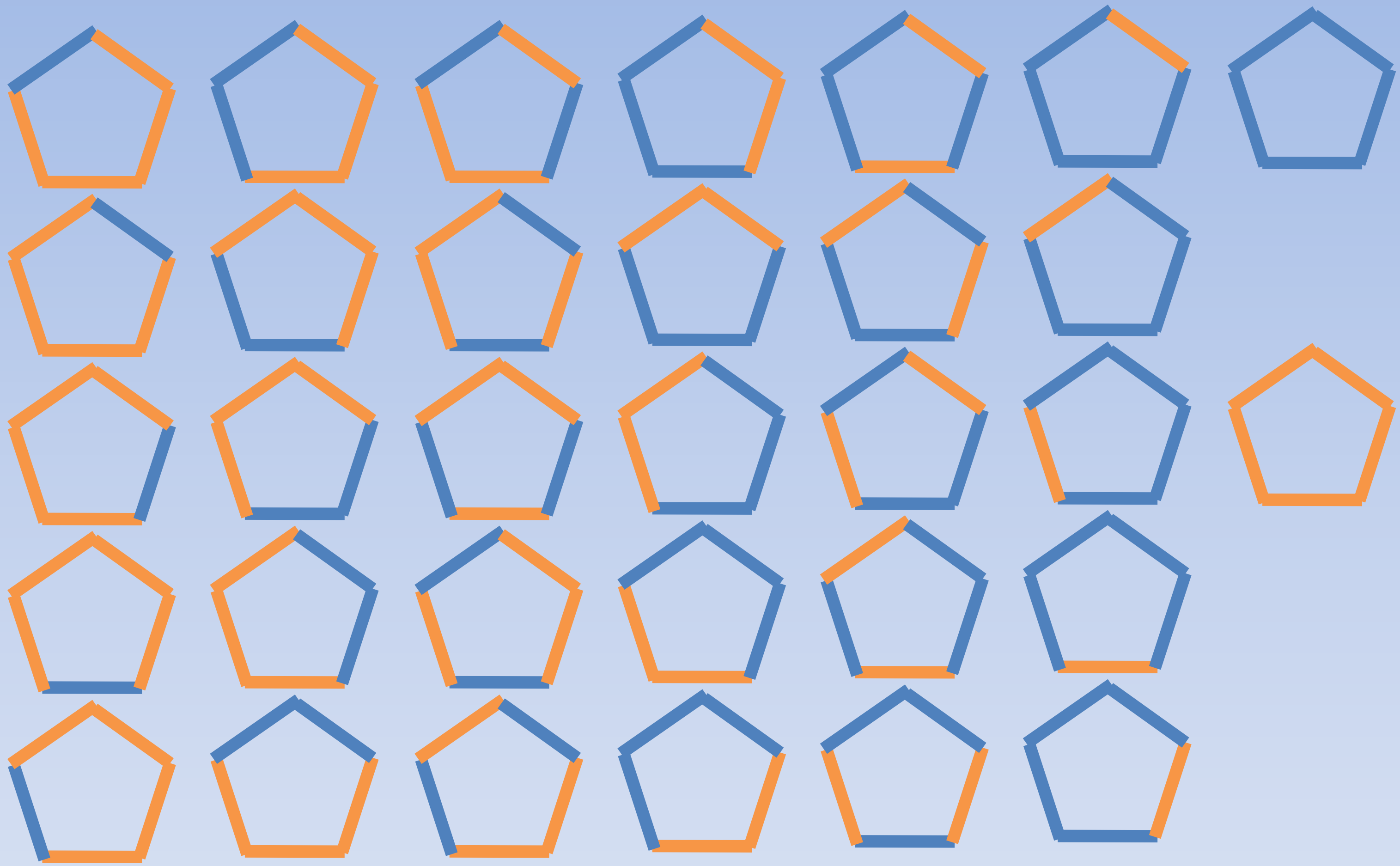

Partitioning these into subsets which are equivalent under rotations shows us the 8 unique tiles.

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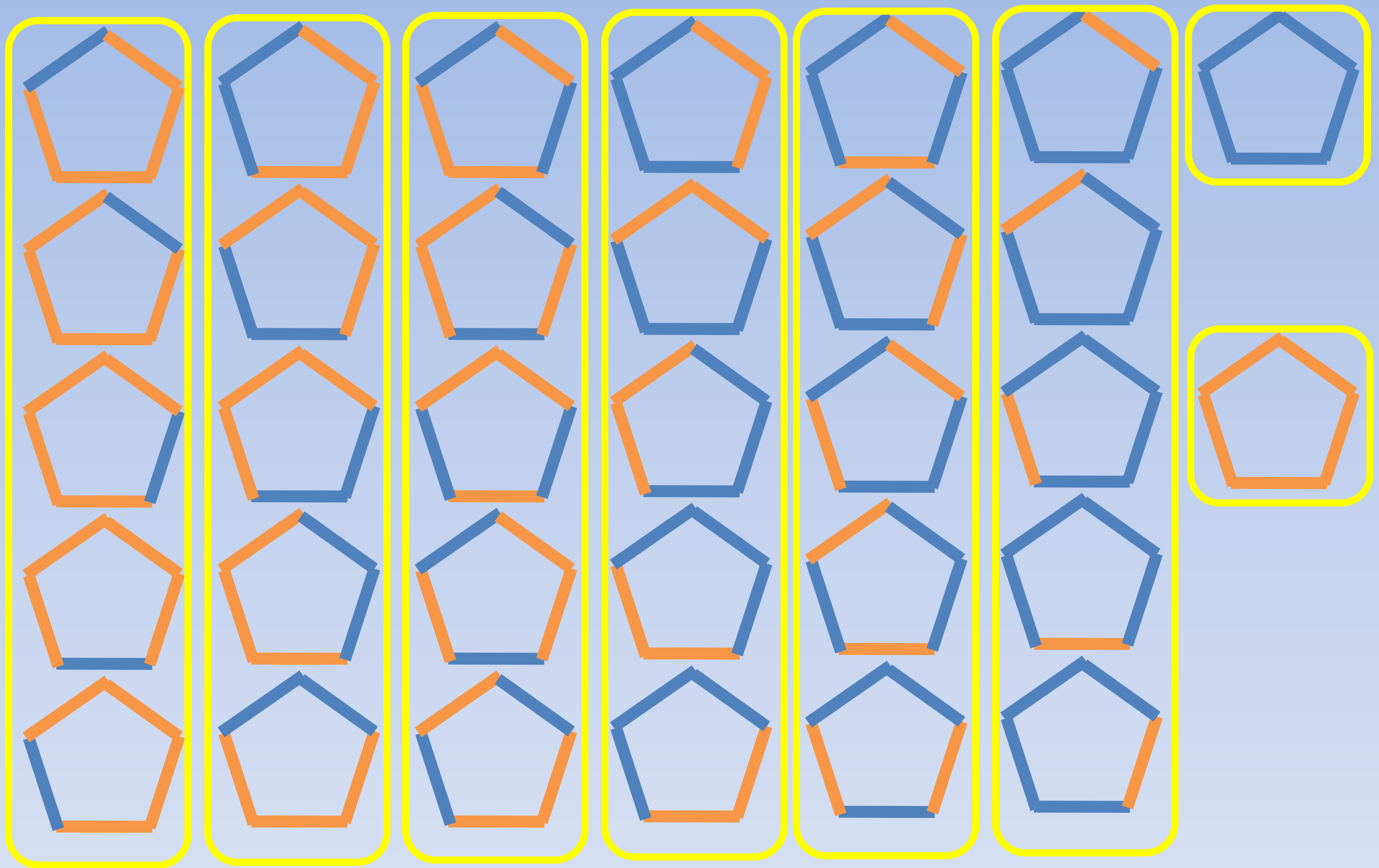

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Just choose one tile from each subset.

Now for some complications…

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[There are actually 130 of them!]

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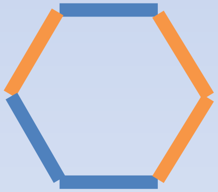

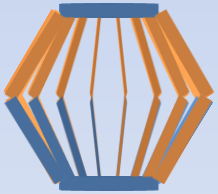

With rotations only, these would not be equivalent.

But with flips, they <u>are</u> considered equivalent.

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Enter: Abstract Algebra!

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Flip vertically

Flip horizontally

Flip over a diagonal

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Flip horizontally
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This set of 8 possible movements of the square is an example of what is known as a ***group***. For our purposes, we will think of a group as "all the ways you could pick up the shape and put it back down again". Each motion is called a ***group element***.

This group of motions of the square is changing the colorings of the square's edges. But not every group element changes every edge coloring…

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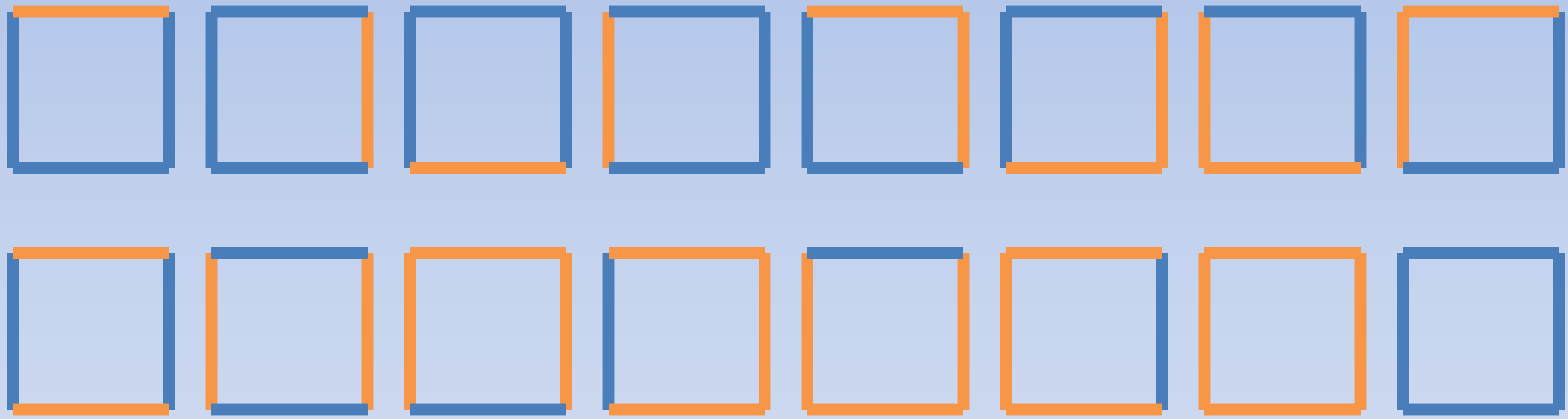
Consider our original example of square tiles and two colors:



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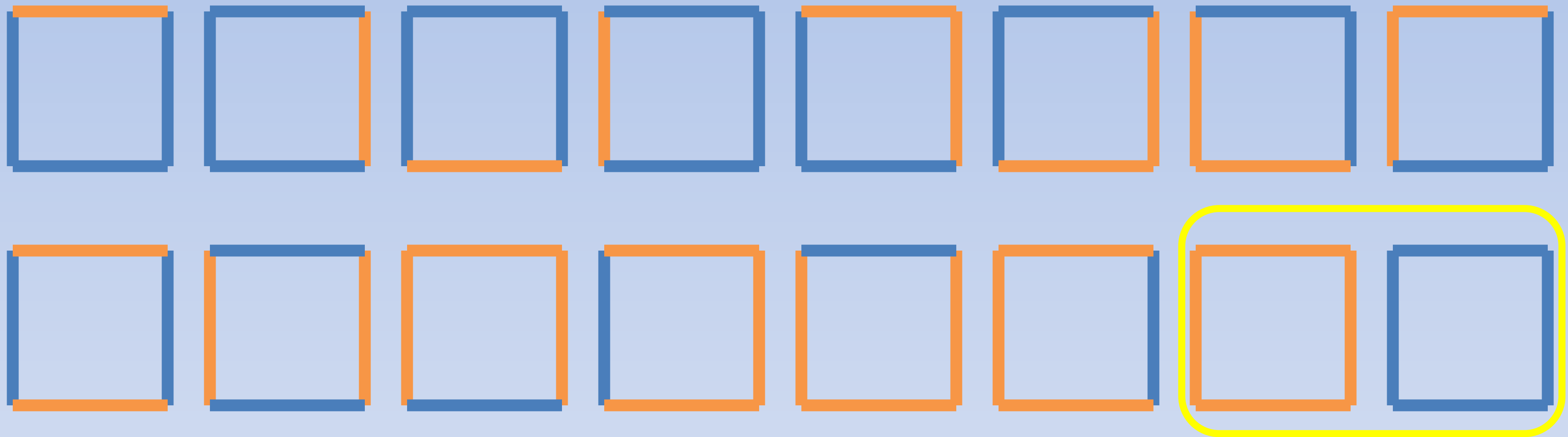

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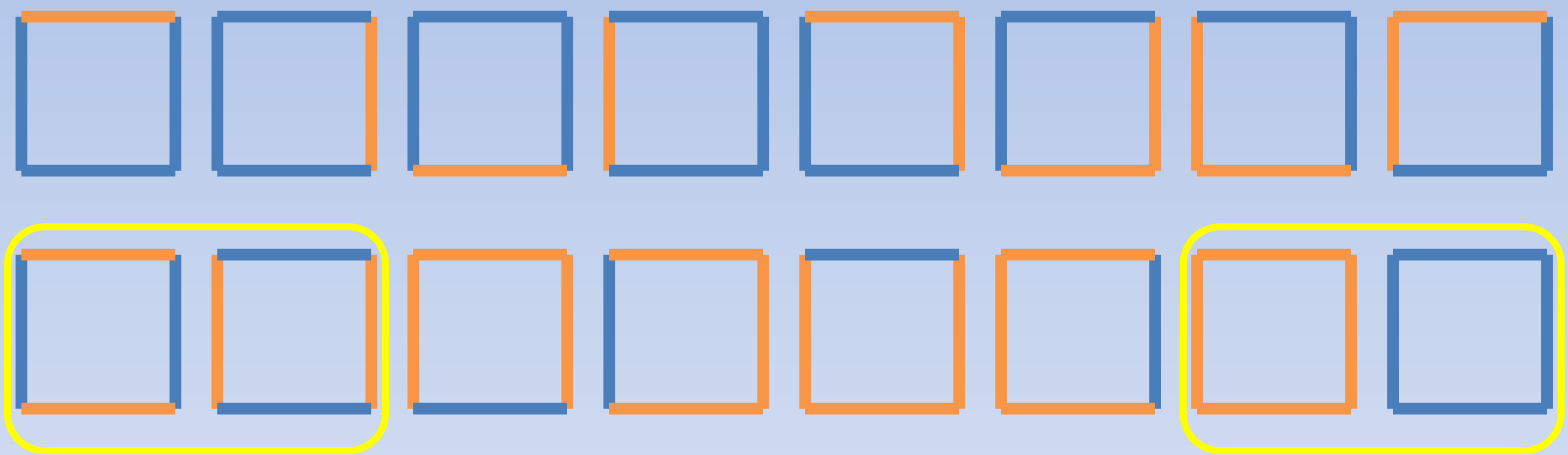

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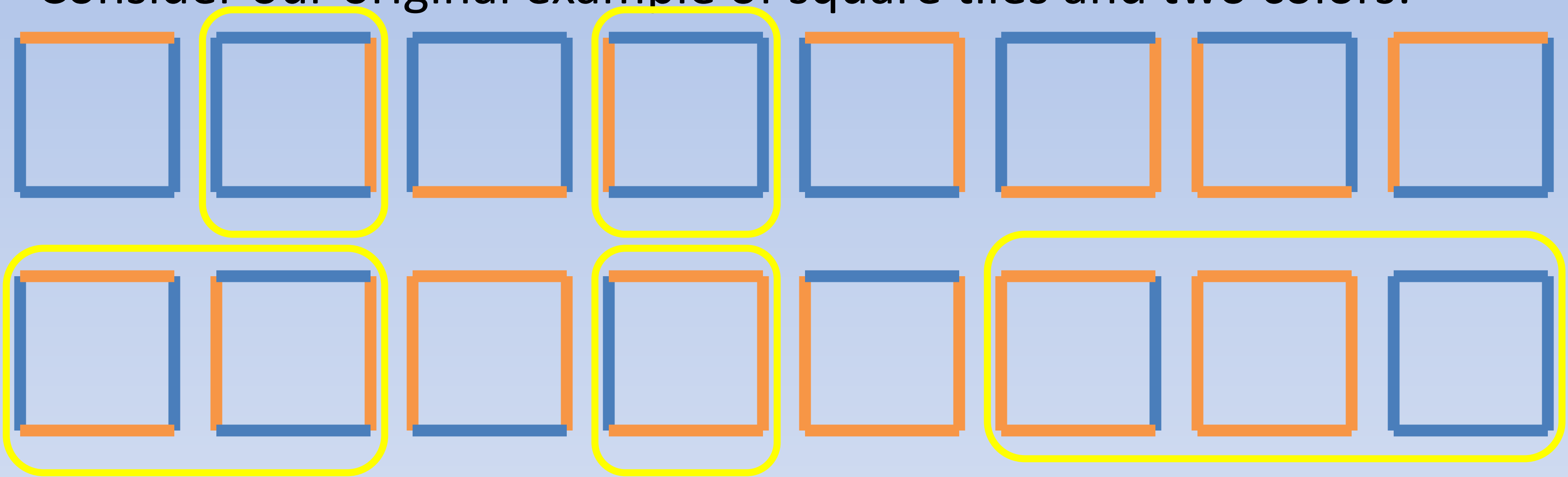

The 90° rotation doesn't change 2 of the colorings. Which ones?
The 180° rotation doesn't change 4 of the colorings. Which ones?
The vertical flip doesn't change 8 of the colorings. Which ones?

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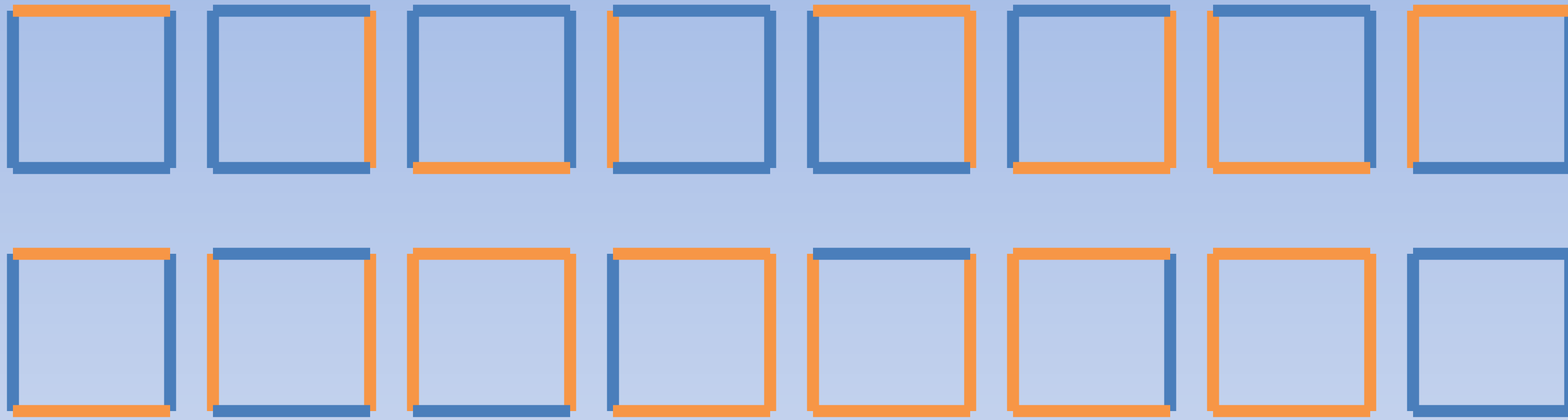

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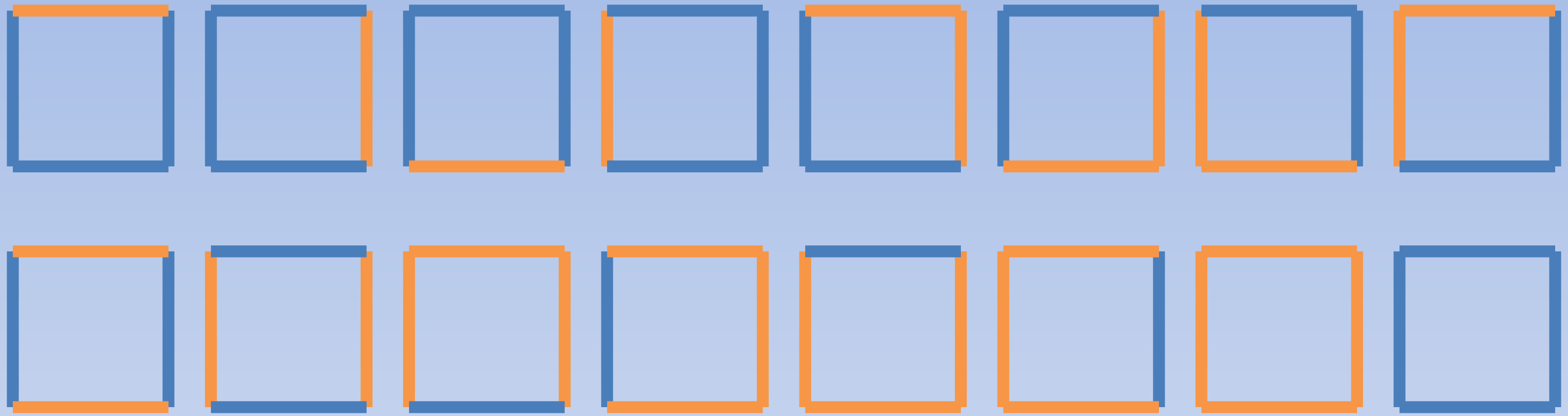

Group element	Number of colorings fixed

Running Total:

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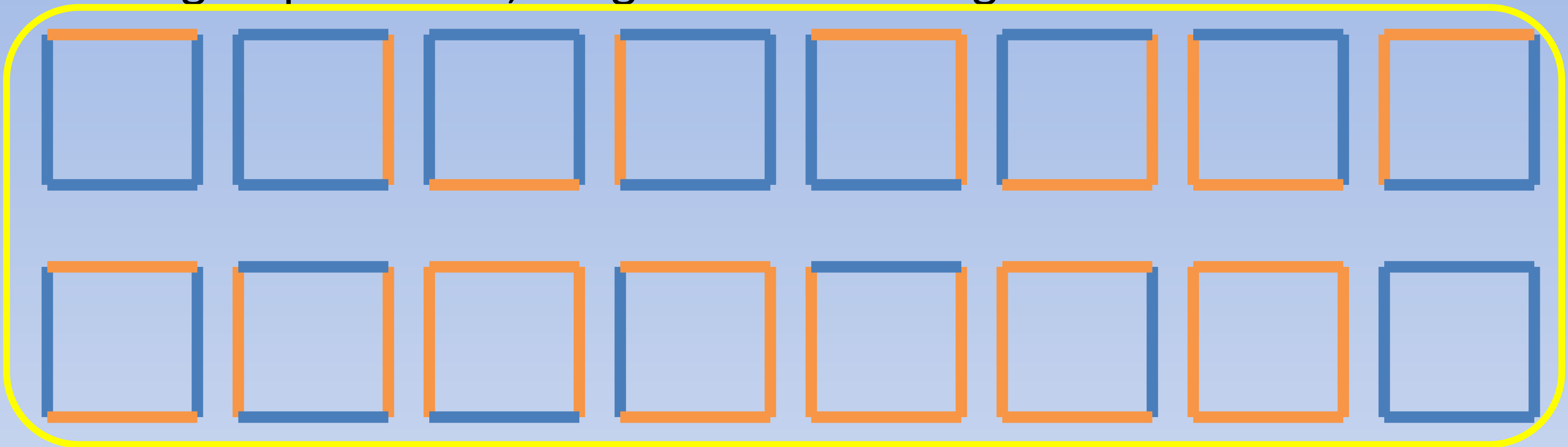

Group element	Number of colorings fixed
0^o Rotation	All of them = 16

Running Total: **16**

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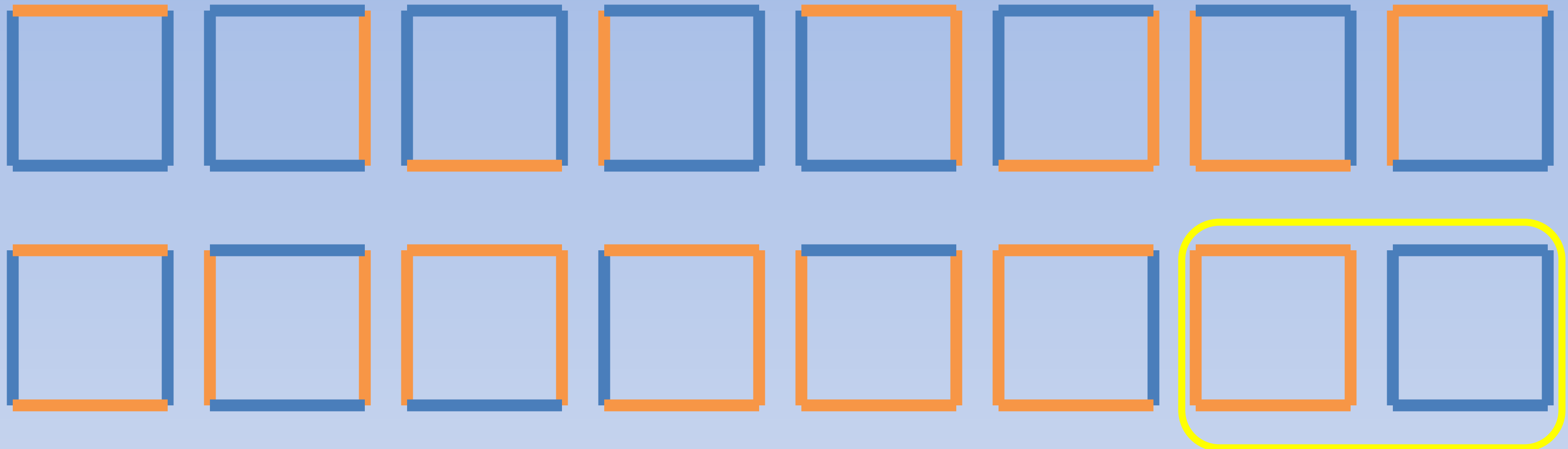

Group element	Number of colorings fixed
90^{o} Rotation	2

Running Total: **16** + **2**

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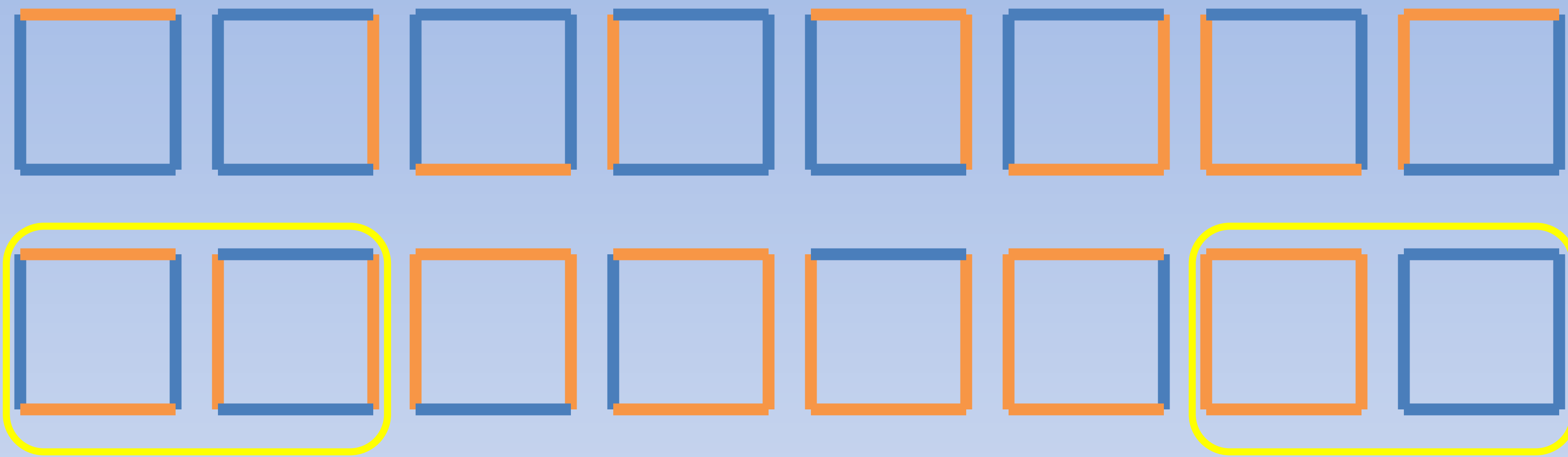

Group element	Number of colorings fixed
180^{o} Rotation	4

Running Total: **16** + **2** + **4**

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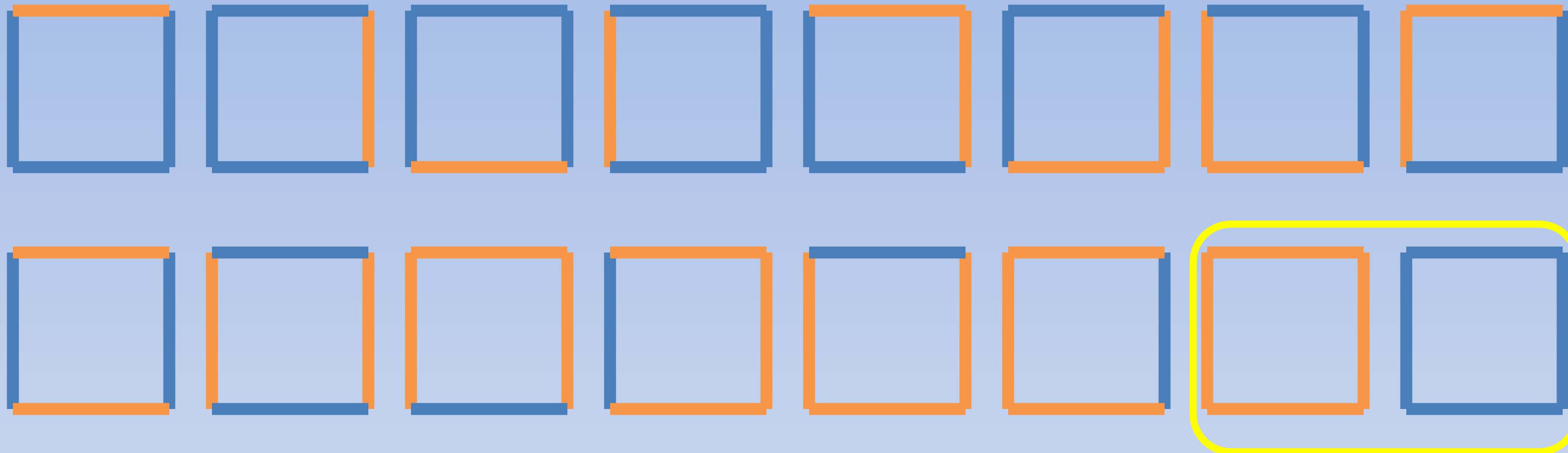

Group element	Number of colorings fixed
270^{o} Rotation	2

Running Total: **16** + **2** + **4** + **2**

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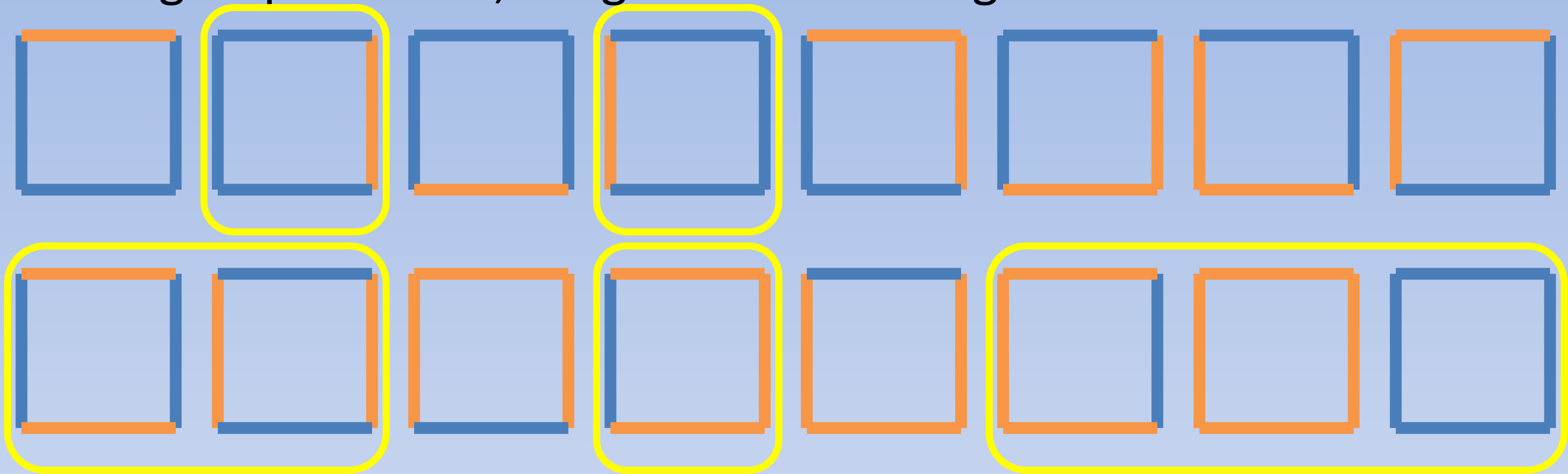

Group element	Number of colorings fixed
Vertical Flip	8

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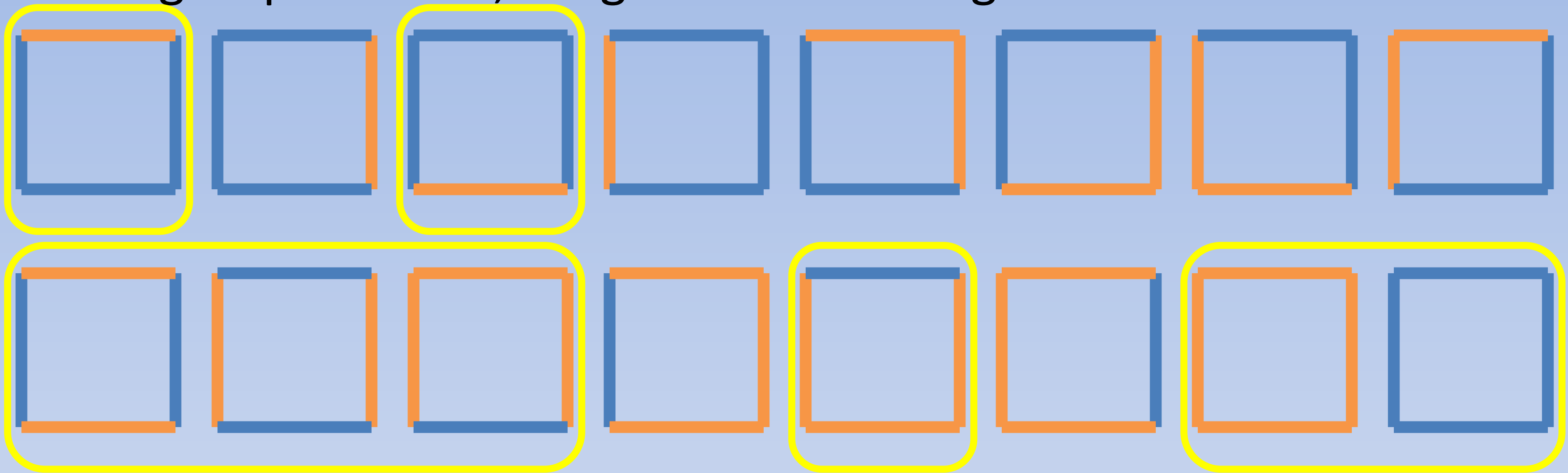

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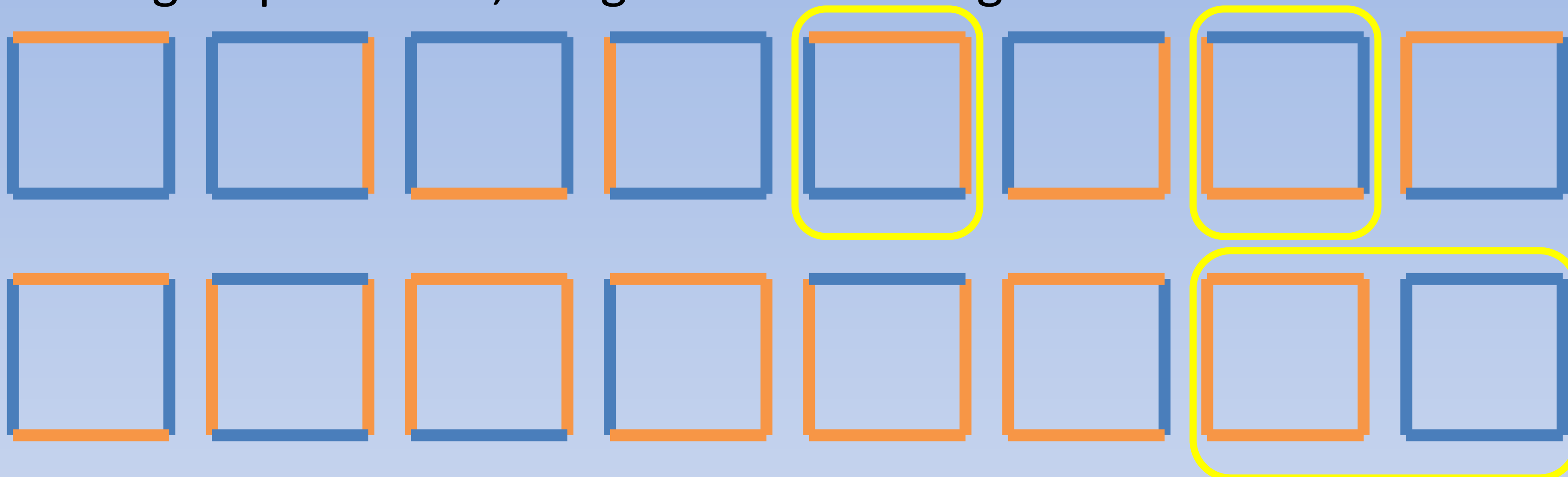

Group element	Number of colorings fixed
Diagonal Flip (over y = x)	4

Running Total: **16** + **2** + **4** + **2** + **8** + **8** + **4**

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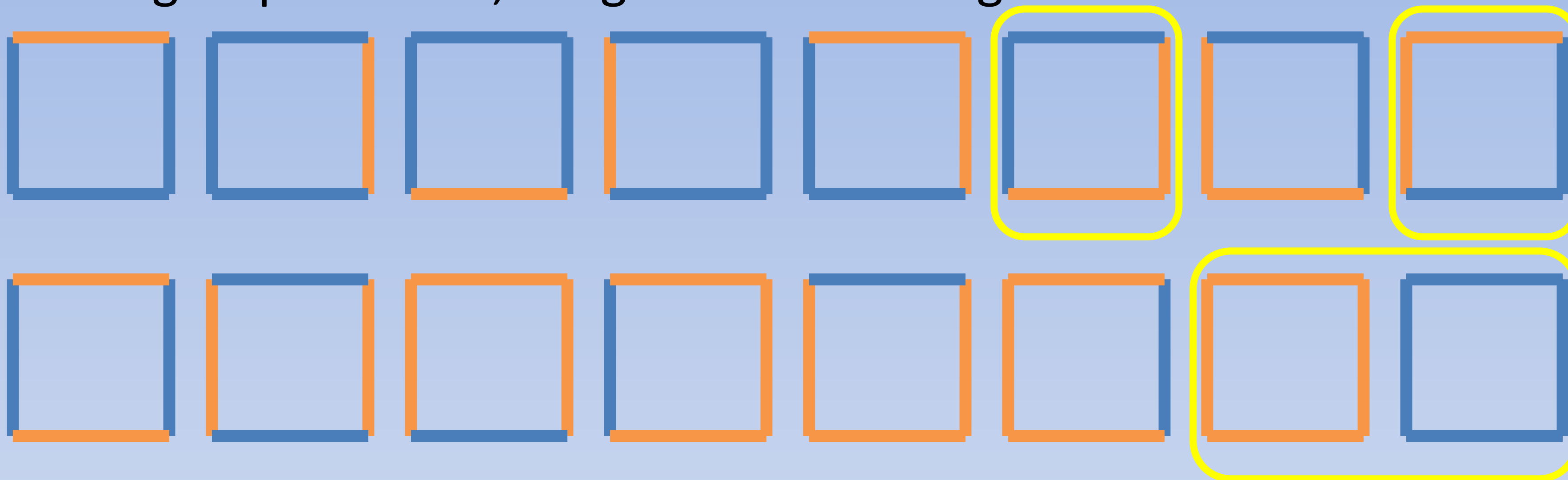

Group element	Number of colorings fixed
Diagonal Flip (over y = -x)	4

Running Total: **16** + **2** + **4** + **2** + **8** + **8** + **4** + **4**

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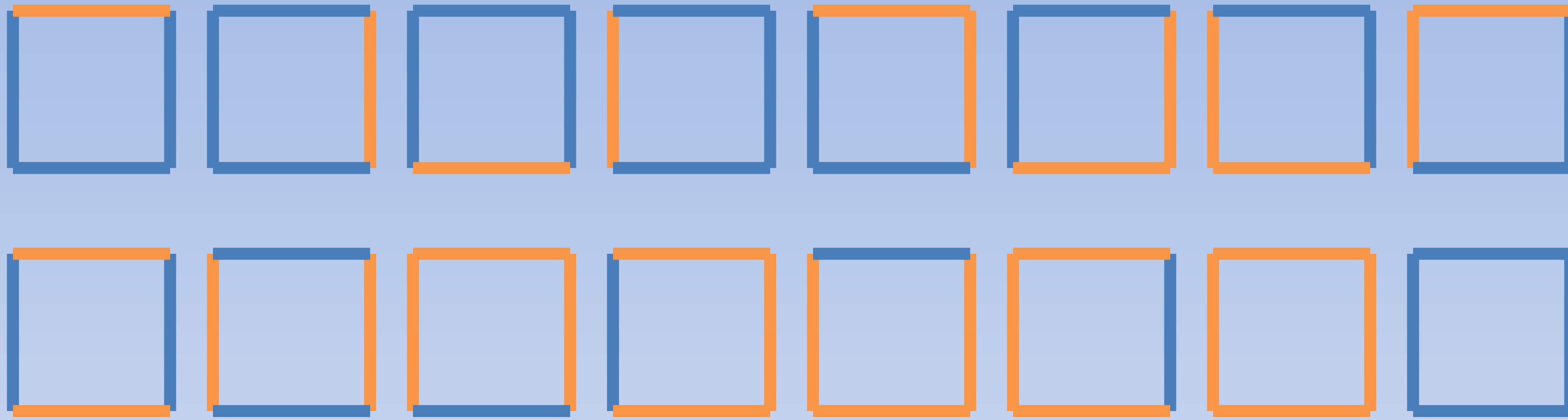

Group element	Number of colorings fixed

Running Total: **16** + **2** + **4** + **2** + **8** + **8** + **4** + **4** = **48**

A famous result in Abstract Algebra says:

$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

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The left side of the equation is the number of ***orbits*** (number of sets of colorings which can be changed into each other by group action). Each orbit is a set of colorings which are all equivalent to each other by some rotation or flip. So, since each orbit represents a unique kind of coloring, the left side says we're counting the number of distinct colorings.

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On the right side of the equation, we have $\frac{1}{|G|}$ which is 1 divided by the number of elements in the group. This is multiplied by a sum whose terms, X^g , are the number of colorings fixed by each group element.

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Notice that we know these numbers:

The group has 8 elements

The sum of colorings fixed by each group element was 48

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Notice that we know these numbers:

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So, the number of distinct colorings is

(1/8)*(48) = **6**

$$\left|{}^{X}\!\!\diagup_{G}\right| = \frac{1}{|G|}\sum_{g \in G} X^{g}$$

This formula is called ***Burnside's Lemma***. Essentially, it says that the number of distinct colorings (orbits) is the average of the sizes of the sets of colorings fixed by group elements.

Some group elements fix a lot of colorings, some only fix a few. This says that the number of colorings fixed by each group element averages out to the number of distinct colorings.

$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

Important point about using the formula:

You can choose which group of motions, *G*, you want to allow.*

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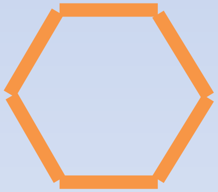

How many rotations are possible?

$$\left| X / G \right| = \frac{1}{|G|} \sum_{g \in G} X^{g}$$

Important point about using the formula:

You can choose which group of motions, *G*, you want to allow.[*]

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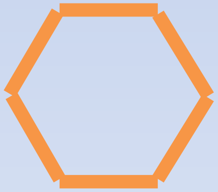
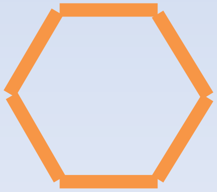


There are 6 possible rotations.

0^{o} 60^{o} 120^{o} 180^{o} 240^{o} 300^{o}

$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

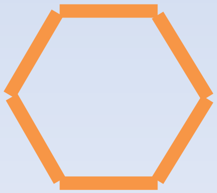
For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.



$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed



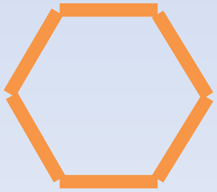
$$\left|X/G\right| = \frac{1}{|G|}\sum_{g \in G} X^{g}$$

For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.

<u>Group element   Colorings left fixed</u>

0^{o} Rotation

Running Total:

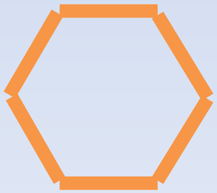


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
0^o Rotation	All 2^6 = 64

Running Total: **64**

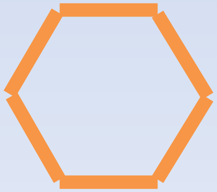


$$\left|X/G\right| = \frac{1}{|G|}\sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
60^o Rotation	

Running Total: **64**

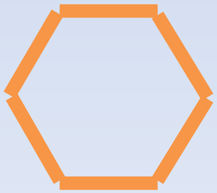


$$\left|X/G\right| = \frac{1}{|G|}\sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
60° Rotation	2

Running Total: **64** + **2**

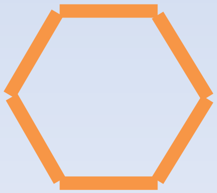


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^{g}$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
120º Rotation	

Running Total: **64** + **2**

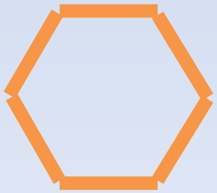


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
120^{o} Rotation	4

Running Total: **64** + **2** + **4**

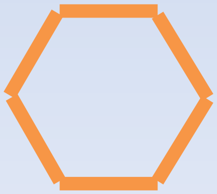


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^{g}$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
180^{o} Rotation	

Running Total: **64** + **2** + **4**

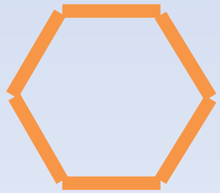


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
180^o Rotation	8

Running Total: **64** + **2** + **4** + **8**

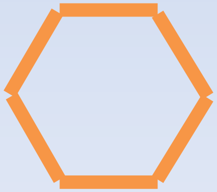


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
240^o Rotation	

Running Total: **64** + **2** + **4** + **8**

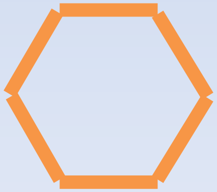


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
240° Rotation	4

Running Total: **64** + **2** + **4** + **8** + **4**

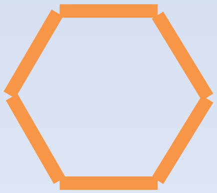


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^{g}$$

For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
300^{o} Rotation	

Running Total: **64** + **2** + **4** + **8** + **4**

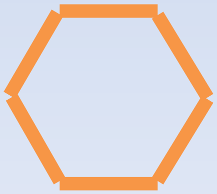


$$\left|X/G\right| = \frac{1}{|G|}\sum_{g \in G} X^g$$

For each rotation, let’s count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed
300^{o} Rotation	2

Running Total: **64** + **2** + **4** + **8** + **4** + **2**

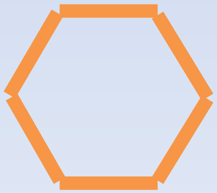


$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

Group element	Colorings left fixed

Running Total: **64** + **2** + **4** + **8** + **4** + **2** = **84**



$$\left|X/G\right| = \frac{1}{|G|}\sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

<u>Group element   Colorings left fixed</u>

Running Total: **64** + **2** + **4** + **8** + **4** + **2** = **84**

Now, determine the number of distinct tiles under rotation.

$$\left| X/G \right| = \frac{1}{|G|} \sum_{g \in G} X^g$$

For each rotation, let's count how many colorings would be left fixed (unchanged). Keep a running total.

<u>Group element Colorings left fixed</u>

Running Total: **64** + **2** + **4** + **8** + **4** + **2** = **84**

Now, determine the number of distinct tiles under rotation.

By the formula, there are (1/6)*(84) = **14** distinct colorings.

How should this number of distinct colorings change if we also allowed flips? Would it go up, down, or stay the same?

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The number should either go down or stay the same, since allowing other motions of the tiles might make some colorings equivalent where they used to be distinct.

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So, how many flips does a hexagon have? Answer: 6

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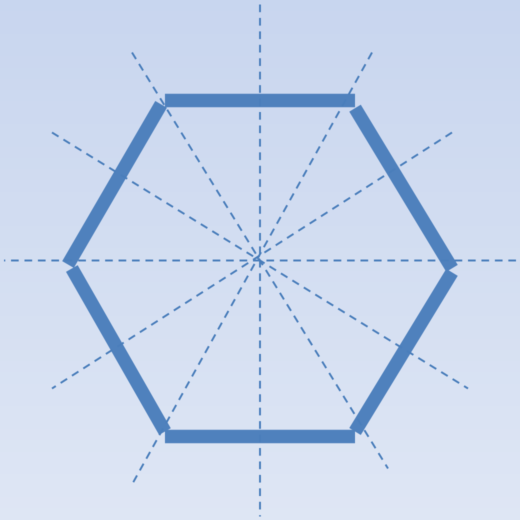
The number should either go down or stay the same, since allowing other motions of the tiles might make some colorings equivalent where they used to be distinct.

So, how many flips does a hexagon have? Answer: 6
You can flip over any of the dotted lines.



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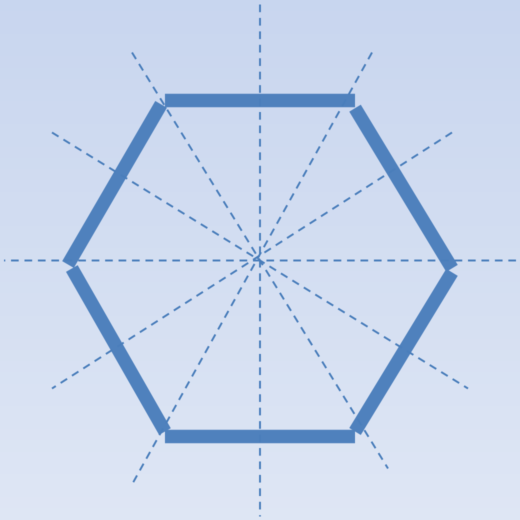
The number should either go down or stay the same, since allowing other motions of the tiles might make some colorings equivalent where they used to be distinct.

So, how many flips does a hexagon have? Answer: 6
You can flip over any of the dotted lines.

We can name the flips by their axis:



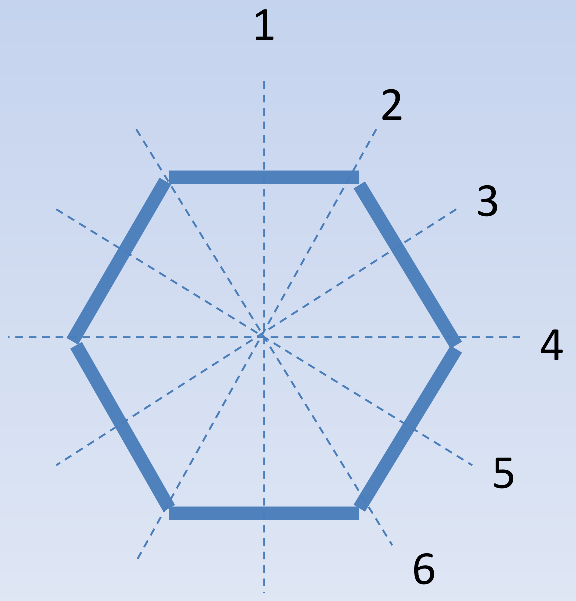
How should this number of distinct colorings change if we also allowed flips? Would it go up, down, or stay the same?

The number should either go down or stay the same, since allowing other motions of the tiles might make some colorings equivalent where they used to be distinct.

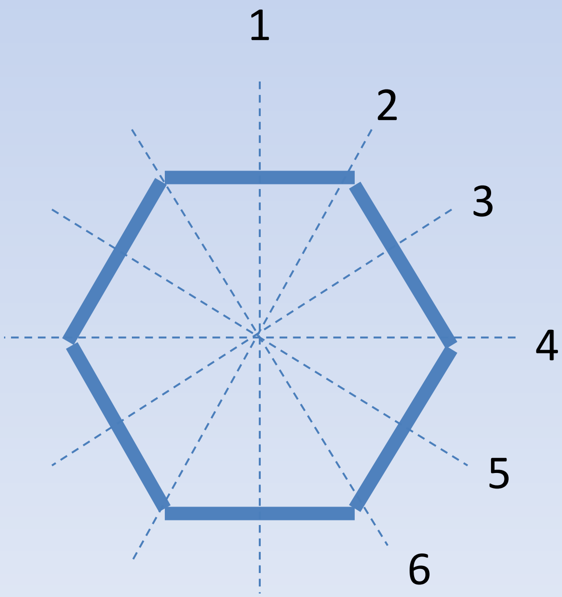
So, how many flips does a hexagon have? Answer: 6
You can flip over any of the dotted lines.

We can name the flips by their axis:

F1, F2, F3, F4, F5 and F6



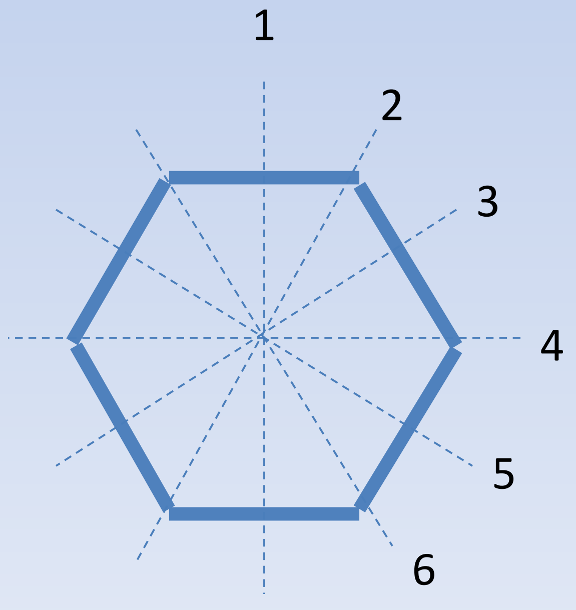
Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

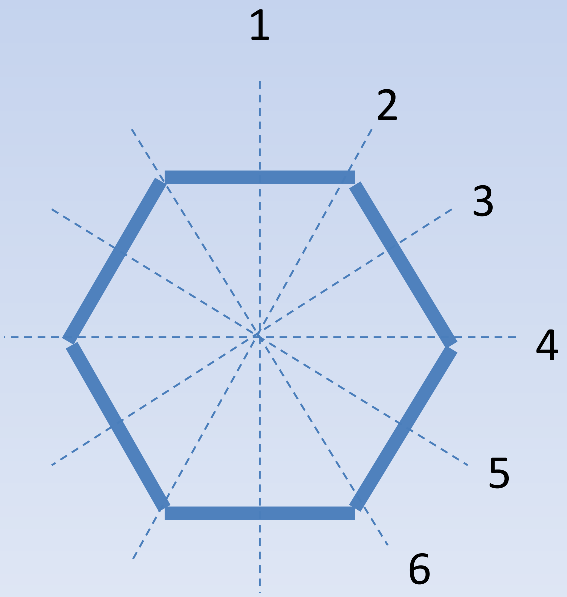
Flips	Number of colorings left fixed

Total:



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

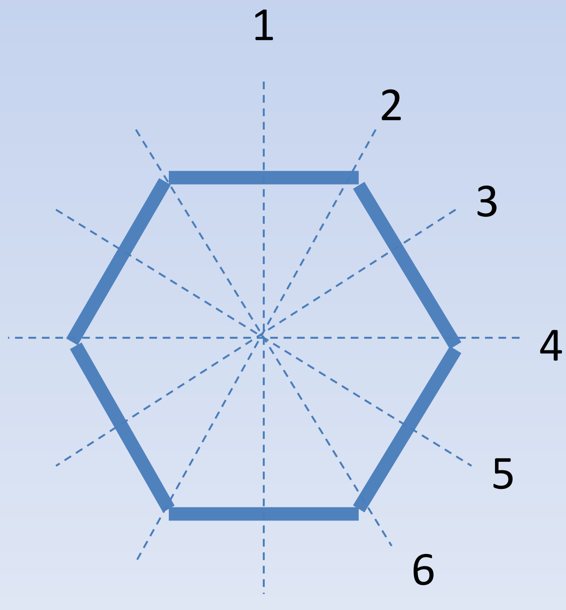
Flips	Number of colorings left fixed
F1	
Total:	



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F1	2*2*2*2 = 16

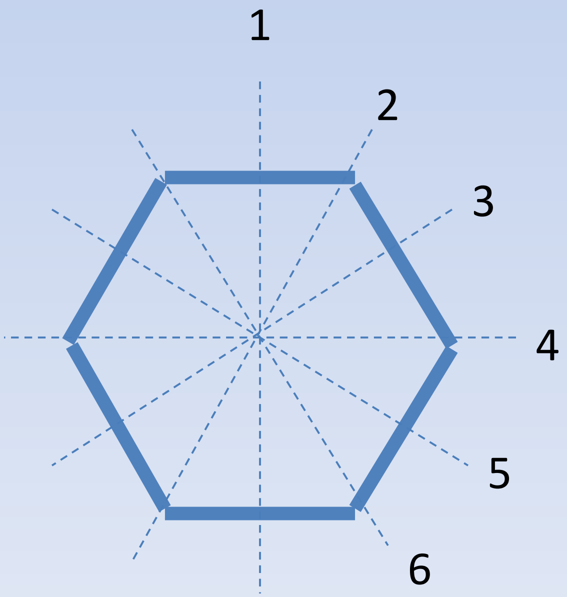
Total: **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F2	

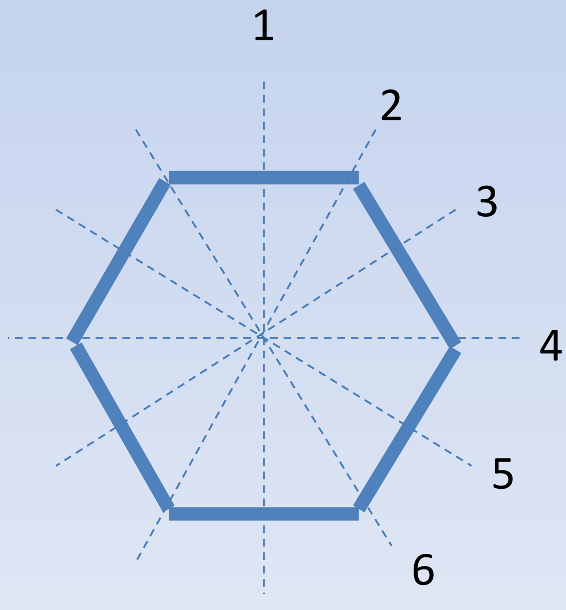
Total: **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F2	2*2*2 = 8

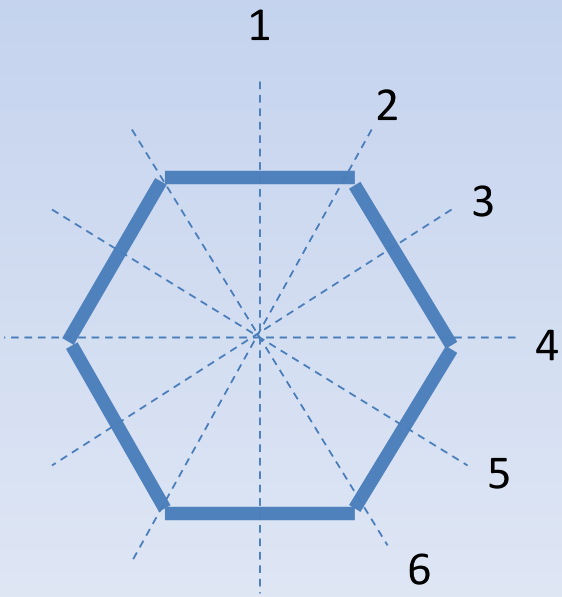
Total: **16** + **8**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F3	

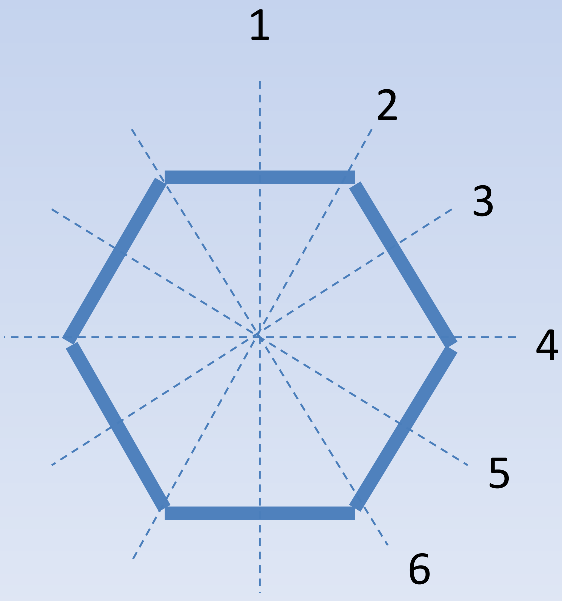
Total: **16** + **8**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F3	2*2*2*2 = 16

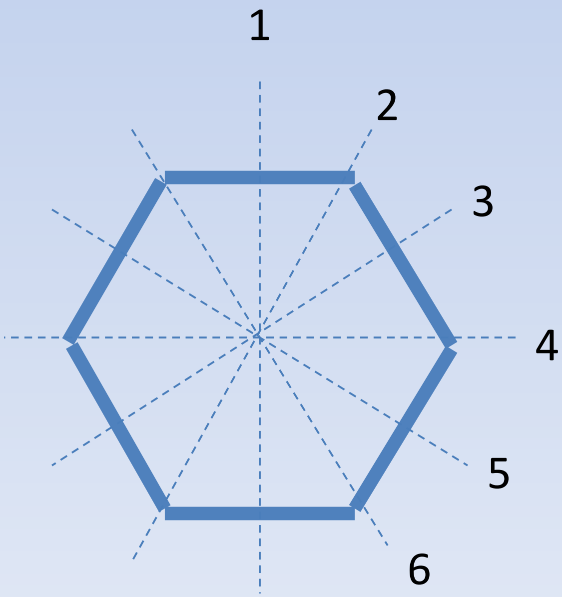
Total: **16** + **8** + **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F4	

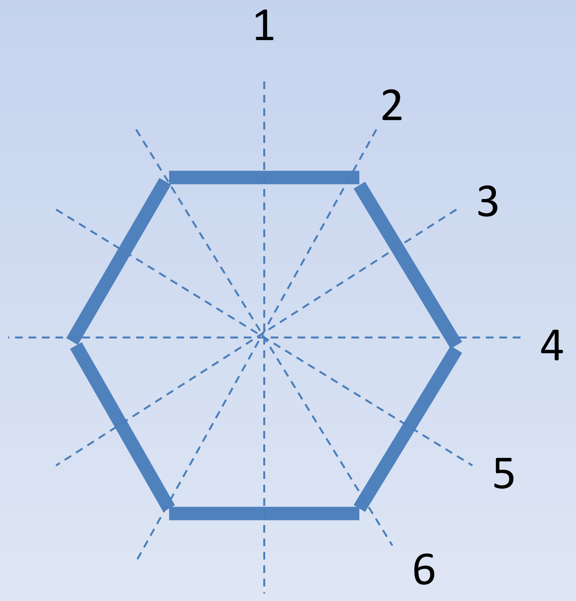
Total: **16** + **8** + **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let’s focus on the flips.

Flips	Number of colorings left fixed
F4	2*2*2 = 8

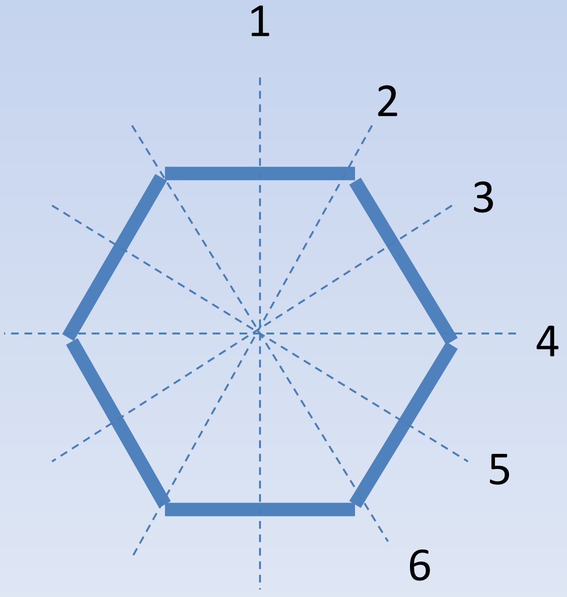
Total: **16** + **8** + **16** + **8**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F5	

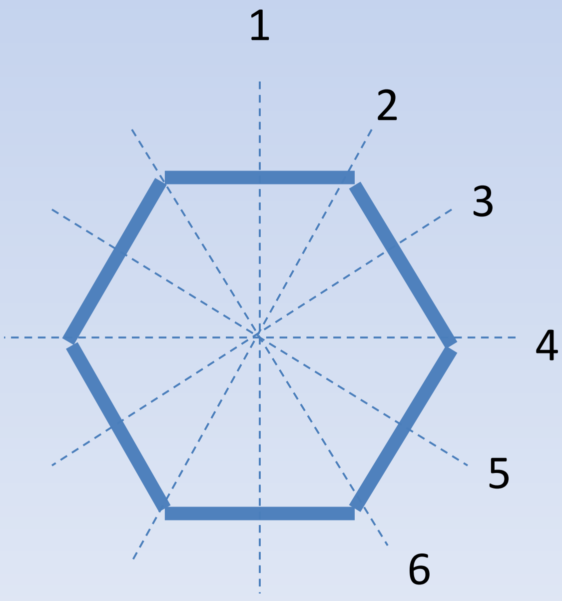
Total: **16** + **8** + **16** + **8**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F5	2*2* 2*2 = 16

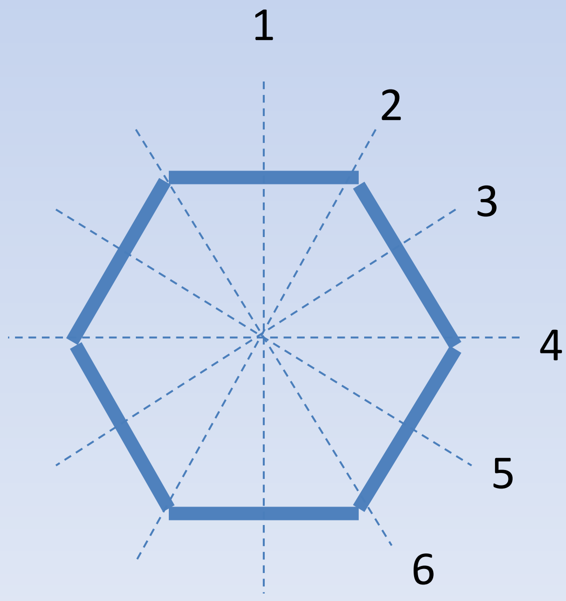
Total: **16** + **8** + **16** + **8** + **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F6	

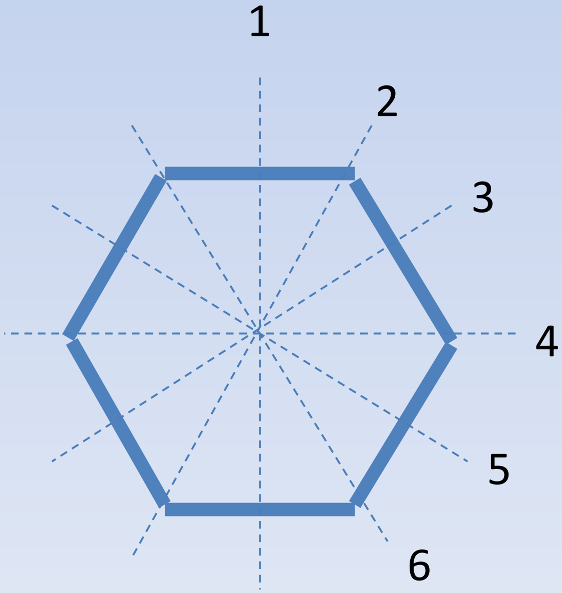
Total: **16** + **8** + **16** + **8** + **16**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips	Number of colorings left fixed
F6	2*2*2 = 8

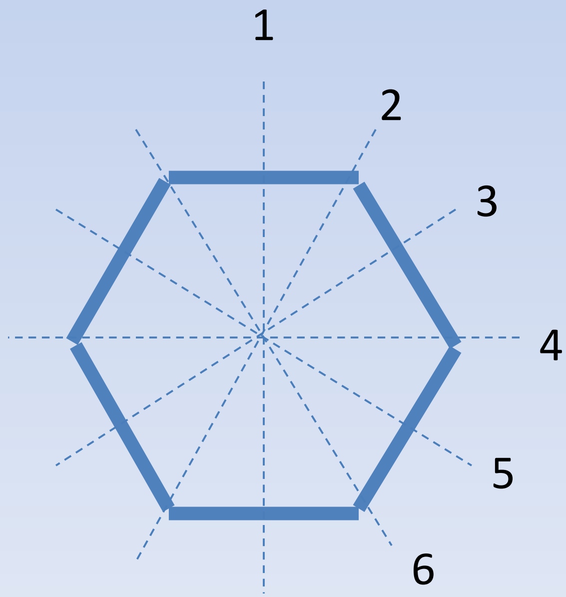
Total: **16** + **8** + **16** + **8** + **16** + **8**



Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

Flips Number of colorings left fixed

Total: **16** + **8** + **16** + **8** + **16** + **8** = **72**



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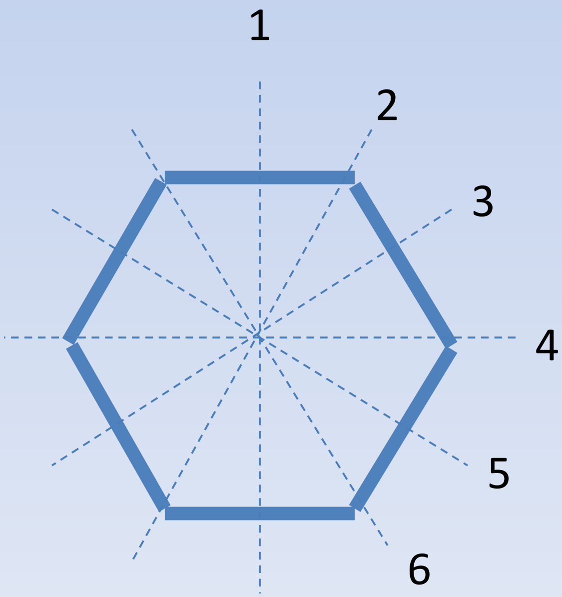
Flips Number of colorings left fixed

Total: **16** + **8** + **16** + **8** + **16** + **8** = **72**

The total fixed by each of the 12 group elements is 72 + 84 = 156



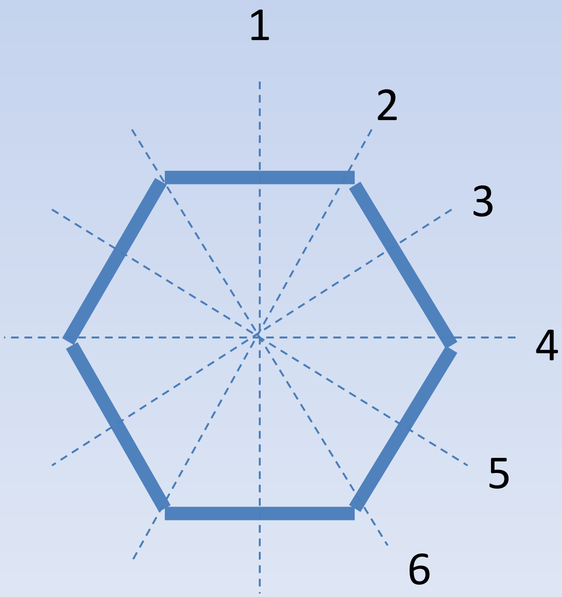
Now we want to count the number of colorings left fixed by each of the 12 group elements (6 rotations and 6 flips). We already counted the number fixed by the rotations, so let's focus on the flips.

<u>Flips   Number of colorings left fixed</u>

Total: **16** + **8** + **16** + **8** + **16** + **8** = **72**

The total fixed by each of the 12 group elements is 72 + 84 = 156
So, the number of distinct colorings where flips and rotations are allowed is:

(1/12)*(156) = 13



Notice that the number of colorings fixed by each group element were just the number of colors (2) raised to some exponent?

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And the exponent was the number of sides which were moved around in a cycle by a group element.

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And the exponent was the number of sides which were moved around in a cycle by a group element.

For example, the 120^o rotation of the hexagon moved the top edge two places clockwise, and moved that edge two more places clockwise, and moved that edge up to the top. This forms a cycle of three edges that must all be the same color if the 120^o rotation leaves the overall coloring fixed.

Notice that the number of colorings fixed by each group element were just the number of colors (2) raised to some exponent?

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Important point: The *cycles* were the key to counting.

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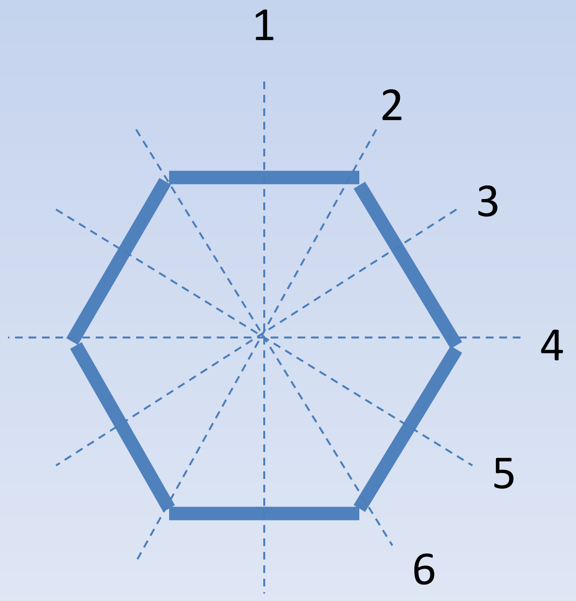

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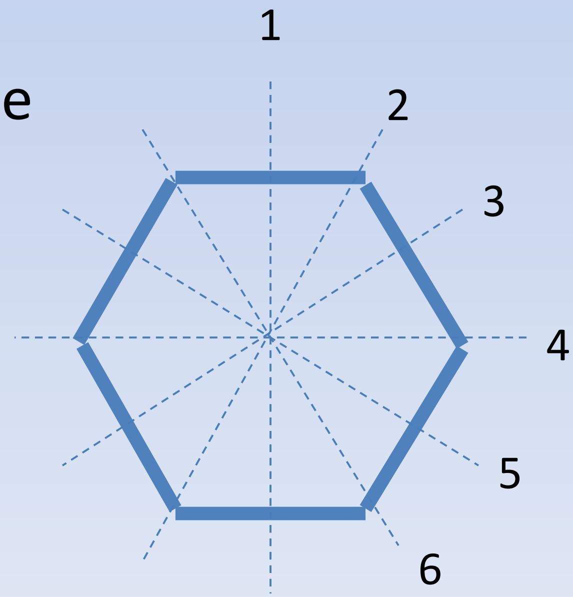

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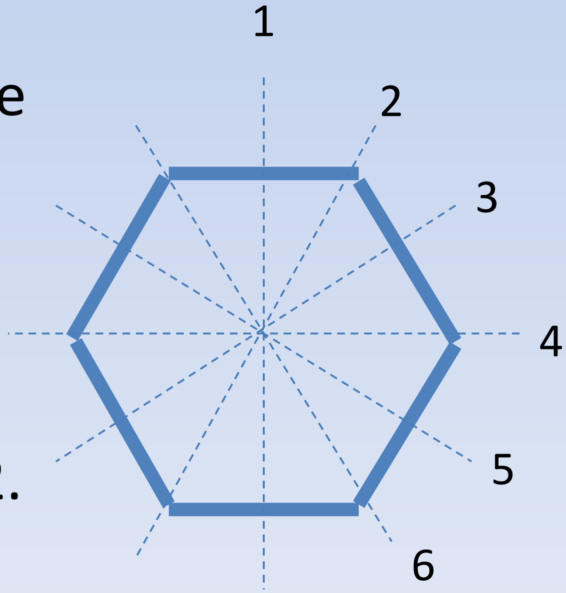
For the flip F1, the cycle index is $x_1^2 x_2^2$ because it has two cycles of length 1 (the top and bottom edges) and two cycles of length 2.



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Here it is for the hexagon:

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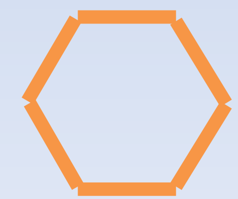

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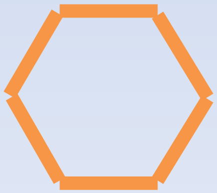
$$\frac{\left(2^{6} + 2(2) + 2(2)^{2} + (2)^{3}\right) + \left(3(2)^{2}(2)^{2} + 3(2)^{3}\right)}{12} = 13$$

Which says that there are 13 distinct hexagon colorings where rotations and flips are allowed.



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Evaluating for $x_* = 2$, we get 10 distinct colorings of a cube.

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Recall: A generating function is a power series whose coefficients encode a sequence.

Examples of generating functions:

The sequence a_n = 1, 1, 1, 1, 1, … is given by the power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

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The sequence b_n = 1, 4, 9, 16, … is given by the power series

$$\frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \ldots$$

The pattern inventory (also called Polya's Enumeration Formula)

For 2-colorings of the edges of an object, say black and white:

In the cycle index, make the substitution $x_j = b^j + w^j$, and expand the polynomial. The coefficient of $w^m b^n$ will count the number of distinct colorings with *m* white edges and *n* black edges.

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More generally, for *p*-colorings, make the substitution $x_j = c_1^j + c_2^j + c_3^j + \ldots + c_p^j$ in the cycle index and expand the polynomial.

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Setting $x_j = b^j + w^j$, we have:

$$\frac{\left((b+w)^6 + 2(b^6+w^6) + 2(b^3+w^3)^2 + (b^2+w^2)^3\right) + \left(3(b+w)^2(b^2+w^2)^2 + 3(b^2+w^2)^3\right)}{12}$$

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Example: Same problem, but this time there are 3 possible colors – black, white and red

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Now we set $x_j = b^j + w^j + r^j$, giving us:

$$\frac{\left((b+w+r)^6+2(b^6+w^6+r^6)+2(b^3+w^3+r^3)^2+(b^2+w^2+r^2)^3\right)+\left(3(b+w+r)^2(b^2+w^2+r^2)^2+3(b^2+w^2+r^2)^3\right)}{12}$$

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Now we set $x_j = b^j + w^j + r^j$, giving us:

$$\begin{aligned}
&= r^6 + r^5(w+b) + 3r^4(w^2 + bw + b^2) + \\
&+ 3r^3(w^3 + 2bw^2 + 2b^2w + b^3) + \\
&+ r^2(3w^4 + 6bw^3 + 11b^2w^2 + 6b^3w + 3b^4) + \\
&+ r(w^5 + 3bw^4 + 6b^2w^3 + 6b^3w^2 + 3b^4w + b^5) + \\
&+ w^6 + bw^5 + 3b^2w^4 + 3b^3w^3 + 3b^4w^2 + b^5w + b^6
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Now we set $x_j = b^j + w^j + r^j$, giving us:

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Again, what does this term tell us?

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$$=r^6+r^5(w+b)+3r^4(w^2+bw+b^2)+$$
$$+3r^3(w^3+2bw^2+2b^2w+b^3)+$$
$$+r^2(3w^4+6bw^3+11b^2w^2+6b^3w+3b^4)+$$
$$+r(w^5+3bw^4+6b^2w^3+6b^3w^2+3b^4w+b^5)+$$
$$+w^6+bw^5+3b^2w^4+3b^3w^3+3b^4w^2+b^5w+b^6$$

Does this term look familiar?

The pattern inventory (also called Polya's Enumeration Formula)

Homework:

1. Consider a necklace with 7 beads. The necklace may be rotated but not flipped, and each bead may be one of 2 colors.

2. How many necklaces with 7 beads can be made with 3 colors?

3. Use Polya's Enumeration Formula to determine how many necklaces with 7 beads and the colors *red, orange, yellow* can be made which have 3 red, 2 orange, and 2 yellow beads.

References:

Applied Combinatorics, 2nd Edition, Alan Tucker, John Wiley and Sons, 1984

Various places on the internet, particularly mathworld.com and Wikipedia.org