On $(P_n; k)$-Vertex Stable Graphs

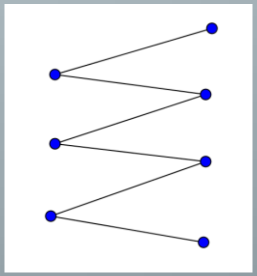
Matthew Ridge

Introduction

- Necessary Information:
  - Graphs of interest:
    - Paths, P_n.
    - Cycles, C_n.
  - Vertex Stable Graphs
    - Minimum stable.
    - Path Stable (Example).
- Interesting things already known.
- Background
  - 501 Speculations and "The BOLD Conjecture".
- 501 Speculations (True/False).
- The_JuggernautV2.
  - How does it work?
- What's Next.

Paths, P_n

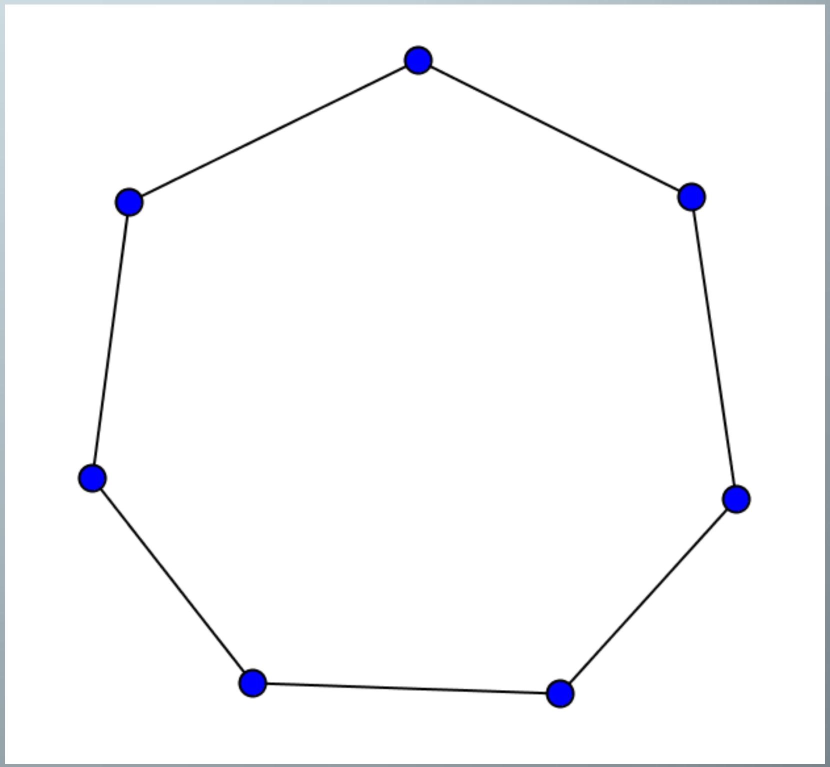
A path is a simple graph on n vertices, whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. The length of a path is $n - 1$.



Graph P_7

Cycles, C_n

A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

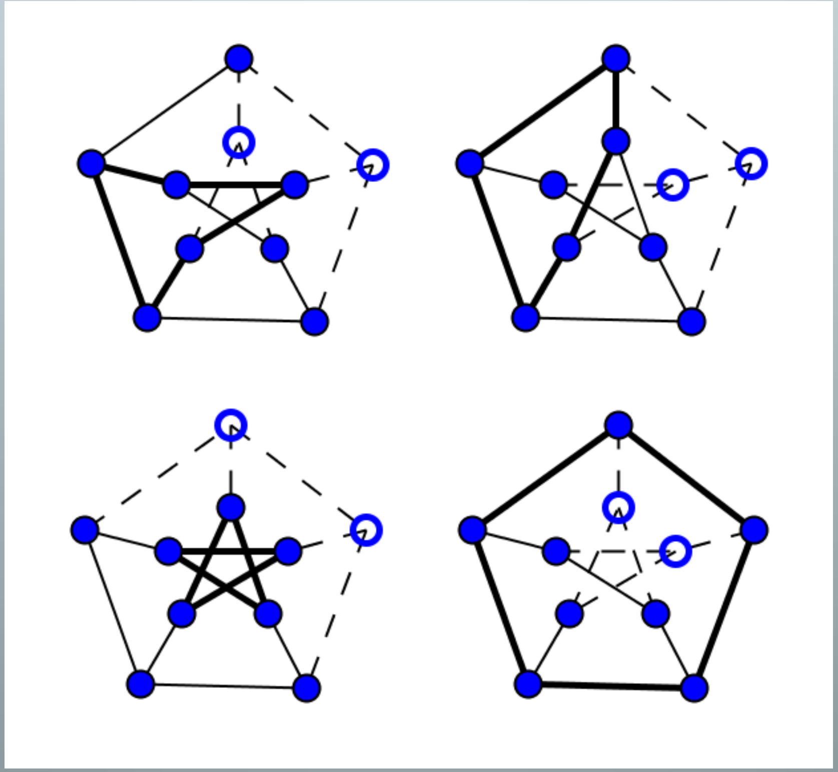


Graph C_7

Vertex Stable Graphs

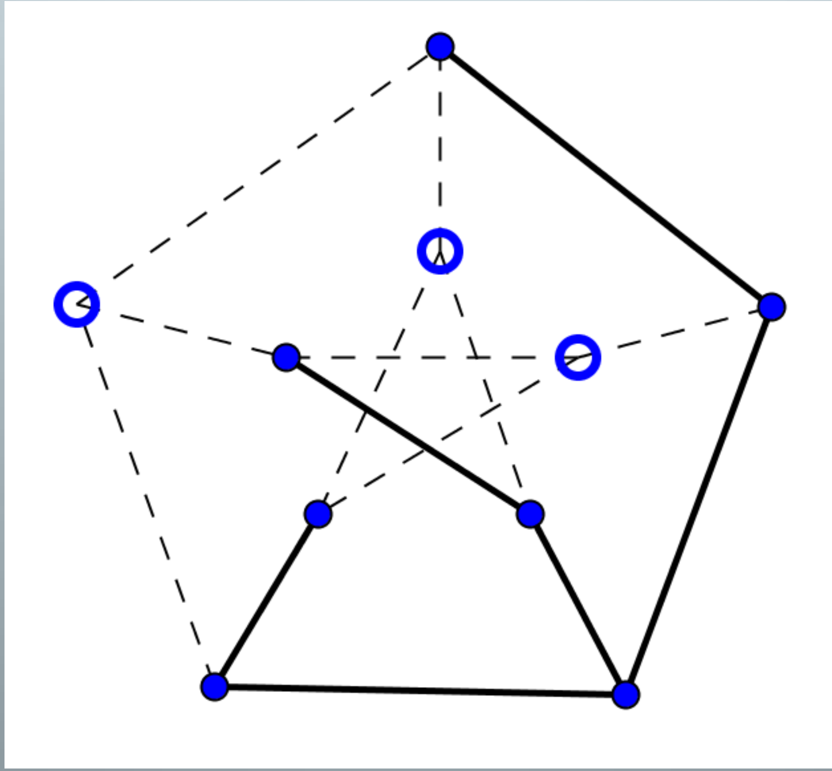
Let H be any graph and k a non-negative integer. A graph G is called $(H;k)$*-vertex stable* or $(H;k)$*-stable* if G contains a subgraph isomorphic to H even after removing ***any*** k of its vertices.

Example of $(H; k)$-stable graph



The Petersen Graph as $(C_5; 2)$-Stable.

Example of Non-stable Graph



The Petersen Graph fails $(C_5; 3)$-Stable.

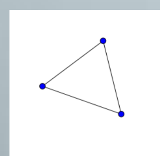
Minimum Stable Graphs

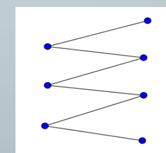
Let the size of graph G be denoted:

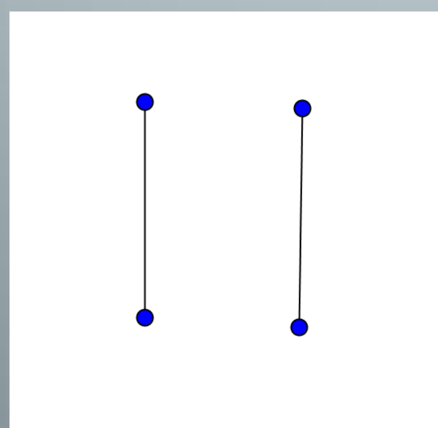
$$\|G\| := |E(G)|.$$

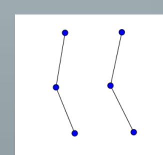
If G has the minimum size of any $(H; k)$-stable graph, then $\|G\| = stab(H; k)$ and we refer to G as a minimum $(H; k)$-stable graph.

Example: minimum $(P_2; 1)$-stable



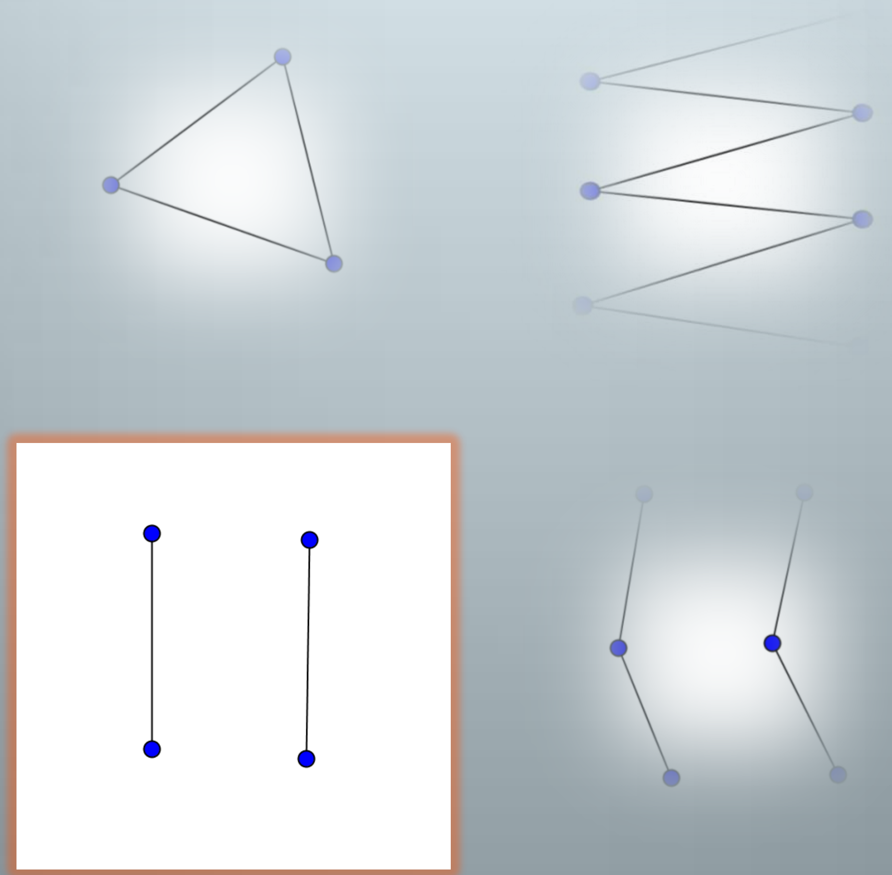






Which one is the minimum $(P_2; 1)$-stable graph?

Example: minimum $(P_2; 1)$-stable



Which one is the minimum $(P_2; 1)$-stable graph?

Interesting things that are already known…

- The Value of $stab(K_q; k)$ and the graphs have been determined with the exception of a few small values of q.[1][3]
- The Value of $stab(C_n; 1)$ has been determined for infinitely many n's (but not all of them!).[4]
- The Value of $stab(K_{n,n}; 1)$ has been determined with $n \geq 2$.[7]
- Lots of upper and lower bounds on the size of the edge sets!

Background

On $\left(K_q; k\right)$-Stable Graphs

An article by Andrzej Żak

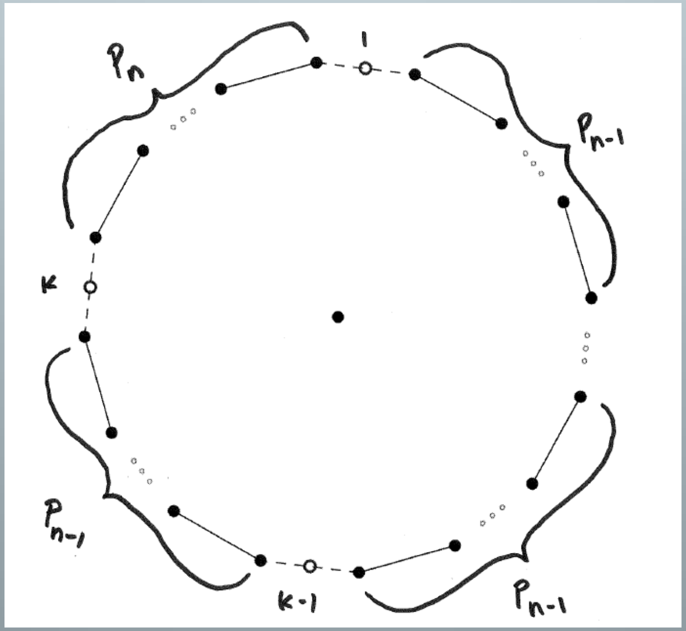
First published online the 15^{th} of October 2012

- Things to think about:
  - Unigueness of minimal stable graph?
  - Families of graphs that guarantee certain stability.
    - C_{kn+1} is $(P_n; k)$-stable.
    - Bipartite graph is $(C_{2n}; k)$-stable.
    - Tripartite graph is $(C_{2n+1}; k)$-stable.
- The BOLD Conjecture.

My (**BOLD**) Conjecture

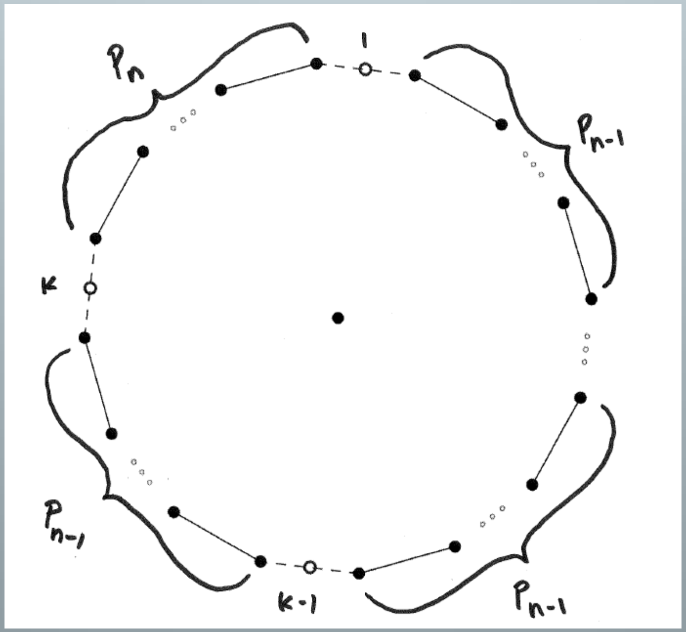
$$stab(P_n; k) = kn + 1$$

The idea:

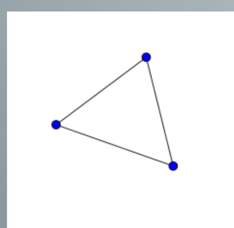


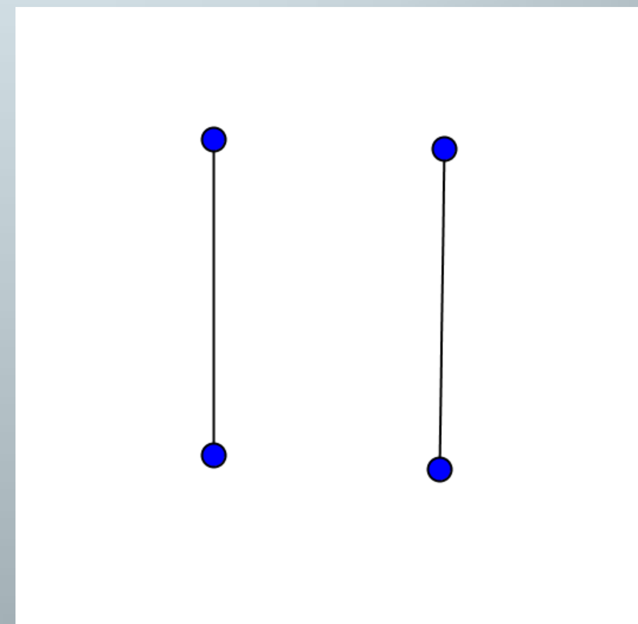
$$\|E(P_n)\| + (k-1)(\|E(P_{n-1})\|) + 2k$$
$$= (n-1) + (k-1)(n-2) + 2k = kn + 1$$

501 Speculations, True/False…



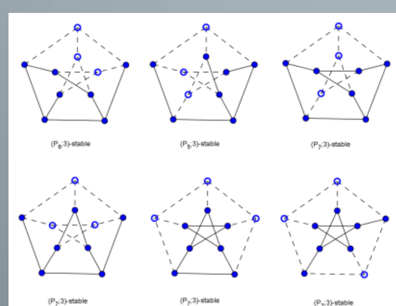
Does the Cycle always win?





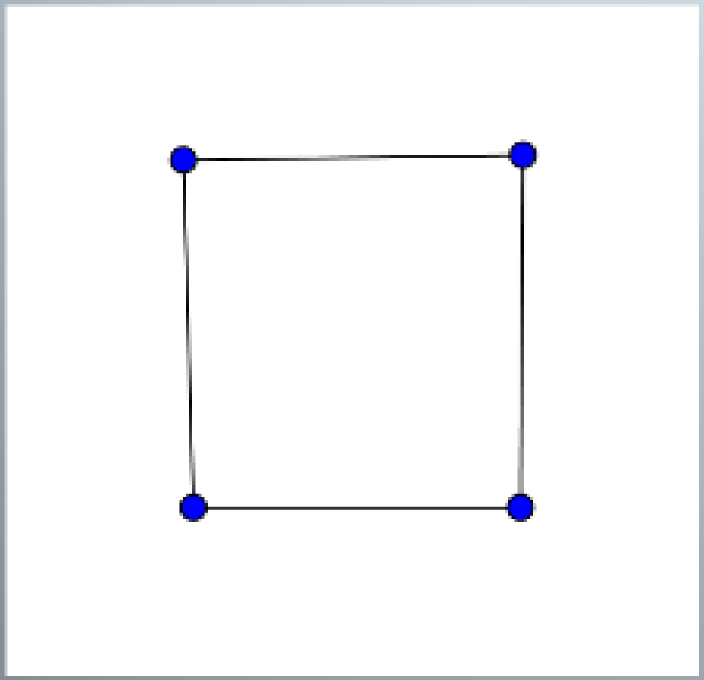
Shamefully it fails on the first case… $n = 2, k = 1$.

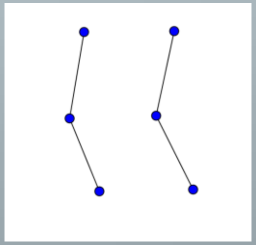
What if we look for only connected solutions???



$$|E(Petersen)| = 15,\ |E(C_{5\cdot 3+1})| = 16$$

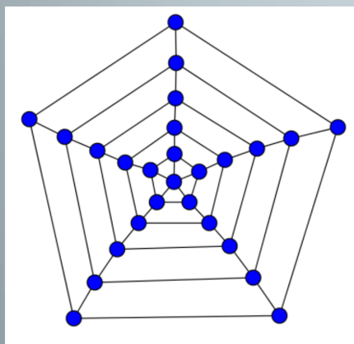
What about Uniqueness?



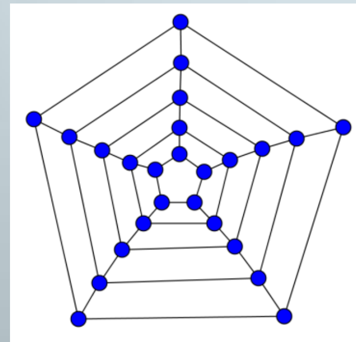


Both are minimum $(P_3;1)$-stable

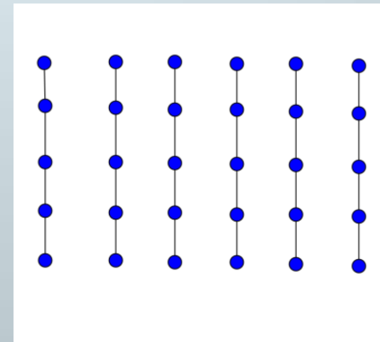
Identifying a pattern.



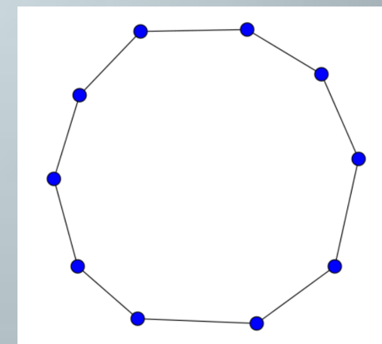
$W_i(n)$



$I_i(n)$



$\mathrm{H}(P_n; j)$

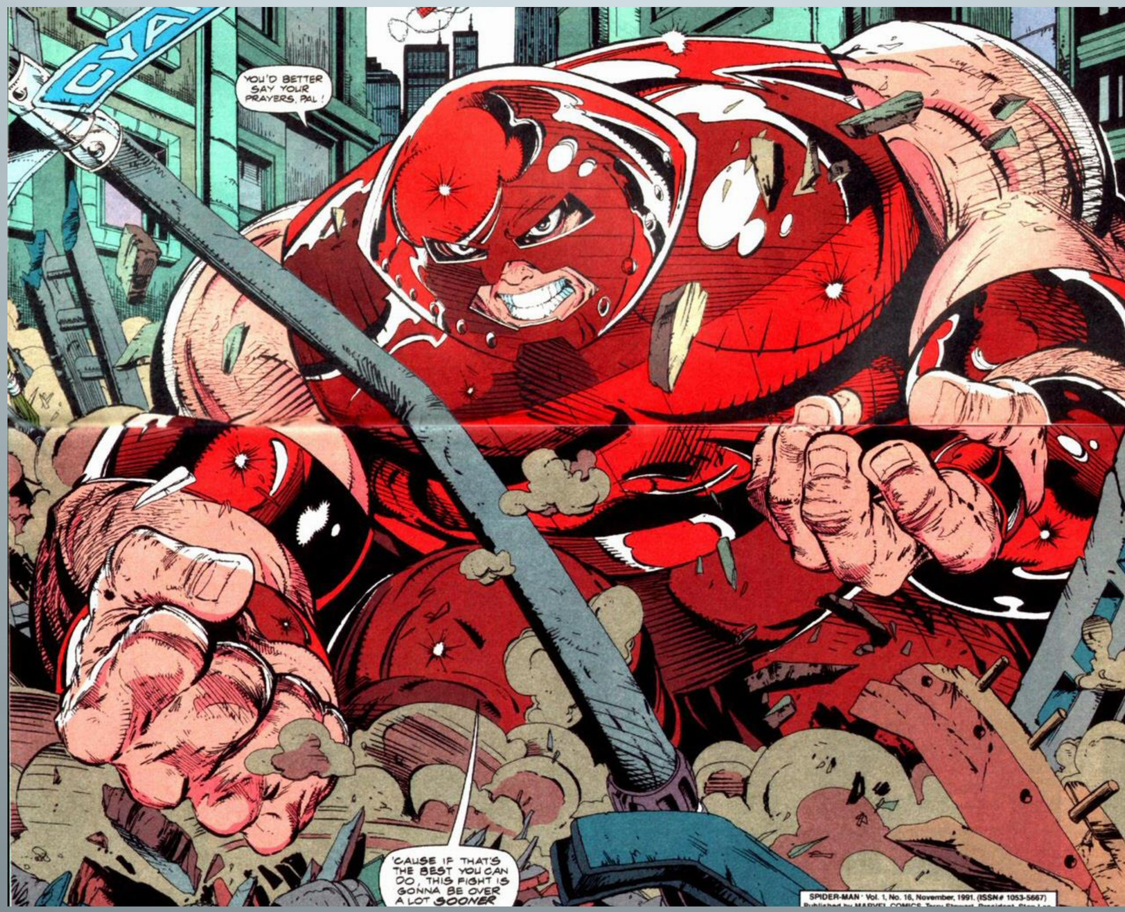


C_n

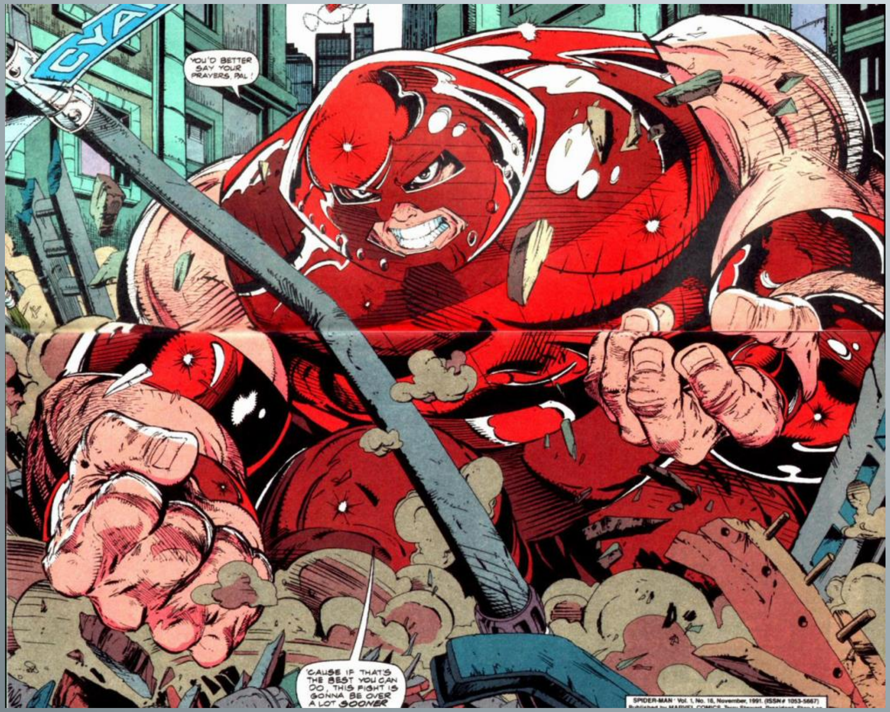
Stability test	Min stable graphs	Stability test	Min stable graphs	Stability test	Min stable graphs	Stability test	Min stable graphs
Min $(P_2;1)$-stable	$H(P_2;2)$	Min $(P_2;2)$-stable	$H(P_2;3)$	Min $(P_2;3)$-stable	?	Min $(P_2;4)$-stable	?
Min $(P_3;1)$-stable	$C_4 \mid H(P_3;2)$	Min $(P_3;2)$-stable	$H(P_3;3)$	Min $(P_3;3)$-stable	?	Min $(P_3;4)$-stable	?
Min $(P_4;1)$-stable	C_5	Min $(P_4;2)$-stable	$C_9 \mid I_1(3) \mid H(P_3;3)$	Min $(P_4;3)$-stable	?	Min $(P_4;4)$-stable	$H(P_4;5)$ |Petersen
Min $(P_5;1)$-stable	C_6	Min $(P_5;2)$-stable	C_{11}	Min $(P_5;3)$-stable	Petersen	Min $(P_5;4)$-stable	?
Min $(P_6;1)$-stable	C_7	Min $(P_6;2)$-stable	C_{13}	Min $(P_6;3)$-stable	?	Min $(P_6;4)$-stable	?
Min $(P_7;1)$-stable	C_8	Min $(P_7;2)$-stable	C_{15}	Min $(P_7;3)$-stable	?	Min $(P_7;4)$-stable	?
Min $(P_8;1)$-stable	C_9	Min $(P_8;2)$-stable	Petersen	Min $(P_8;3)$-stable	?	Min $(P_8;4)$-stable	?
Min $(P_9;1)$-stable	C_{10}	Min $(P_9;2)$-stable	?	Min $(P_9;3)$-stable	?	Min $(P_9;4)$-stable	?

How does one turn guesses into fact?

The_JuggernautV2

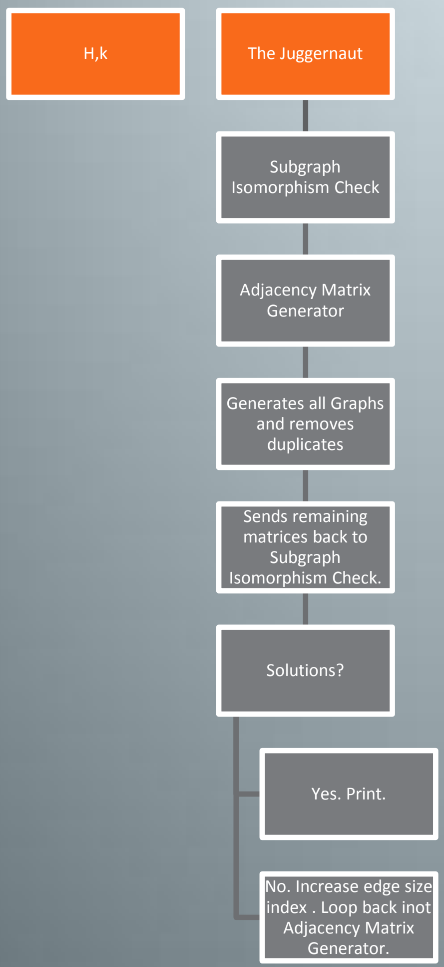


The_JuggernautV2



The Juggernaut is a MATLAB program that is designed to search for minimum stable graphs for any graph H, and non-negative integer k, by generating all possible graphs and destroying each one until finds the solution. In other words: The brute force method of finding solutions…

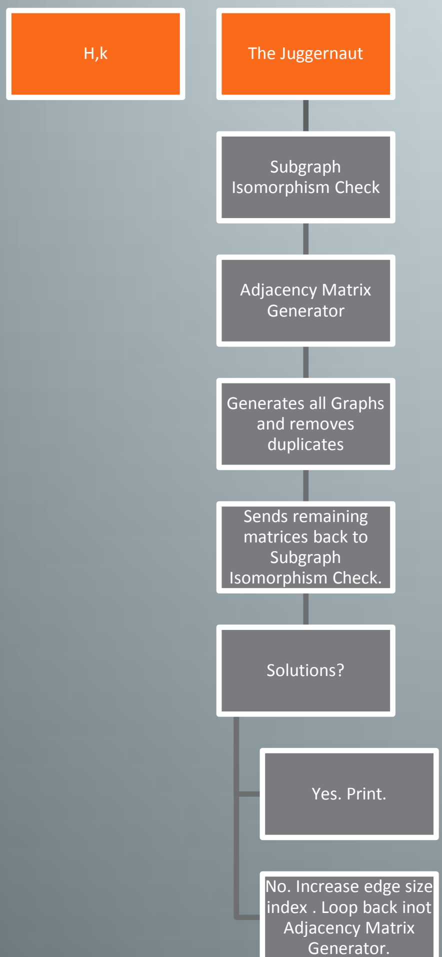
How does it work?



You start with first creating an adjacency matrix, H, and picking vertex removal set size, k. These two values are what The_JuggernautV2 want.

Example:
Let $H = [0\ 1\ 0; 1\ 0\ 1; 0\ 1\ 0],\ k = 1.$
The program will look for the minimum $(P_3; 1)$-stable graph.

How does it work?

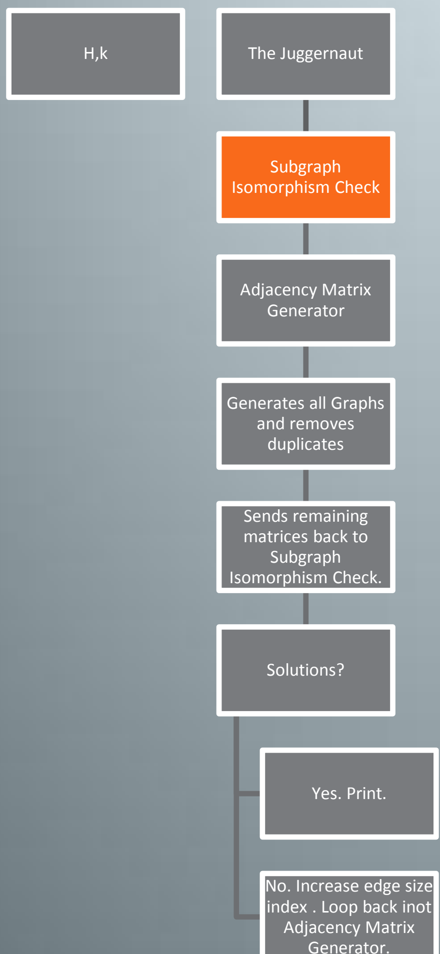


An upper bound on edges of G is set based off of the trivial case: $(k+1)||H||$.

And a lower bound is set based on $||H||+k$.

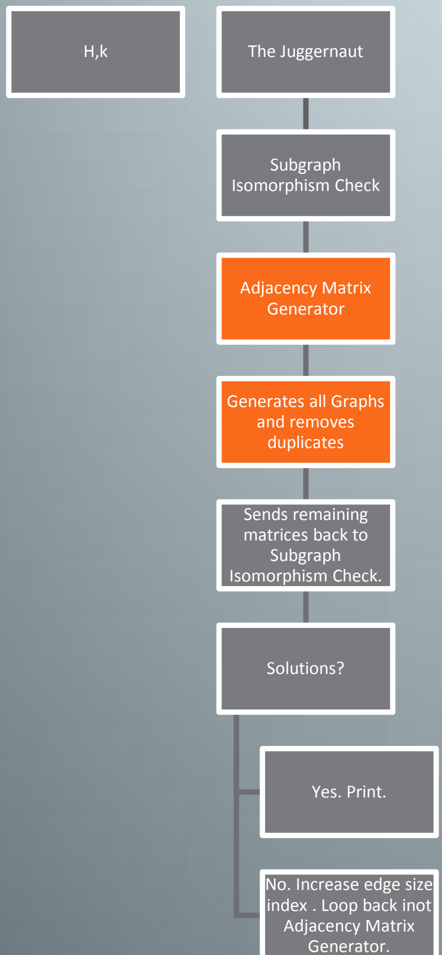
These bounds are set as indices for the primary loop.

How does it work?



The Subgraph Isomorphism Check takes in the indexed edge size, e, and H. It sends e to the Adjacency Matrix Generator.

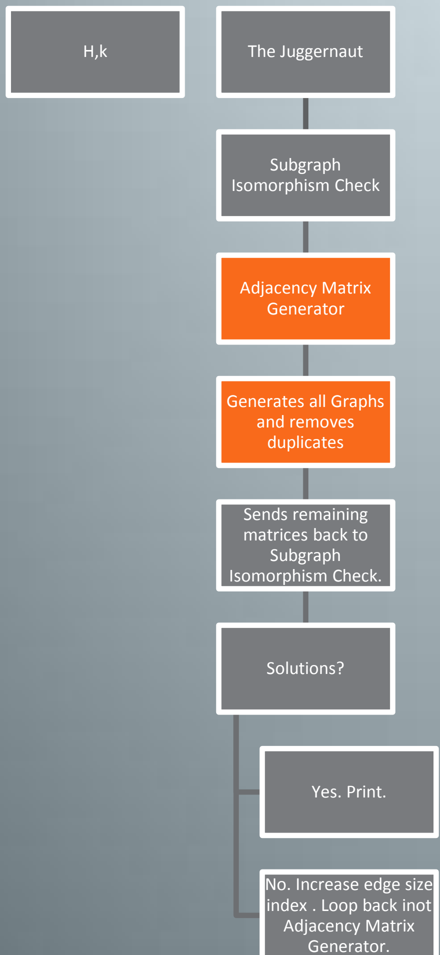
How does it work?



The Adjacency Matrix Generator indexes the upper diagonal of the matrix and stores the locations in a list.

$$\begin{bmatrix} 0 & 1\cdot n+1 & 2\cdot n+1 & & (n-1)\cdot n+1 \\ & 0 & 2\cdot n+2 & \cdots & (n-1)\cdot n+2 \\ & & 0 & & (n-1)\cdot n+3 \\ & & & \ddots & \vdots \\ & & & & (n-1)\cdot n+(n-1) \\ & & & & 0 \end{bmatrix}$$

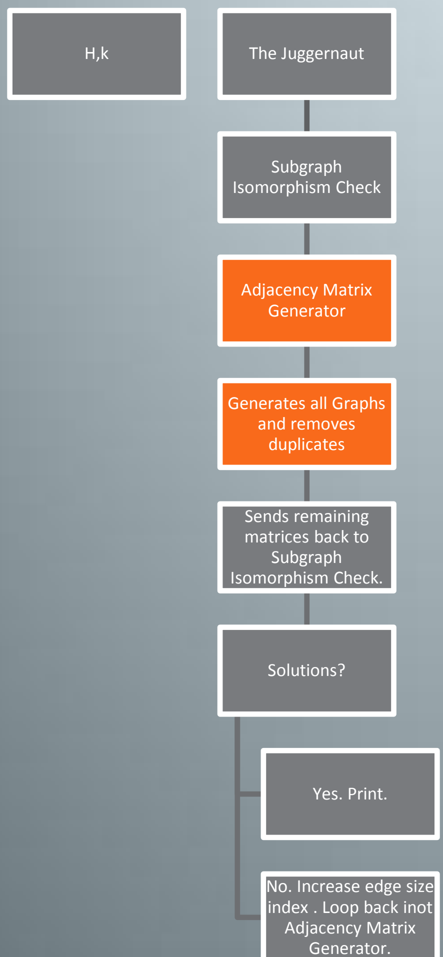
How does it work?



From this list, e locations are chosen. A standard choose function generates all of the possible permutations.
A value of 1 is mapped to all e of the chosen locations, and 0 to the remaining spots.

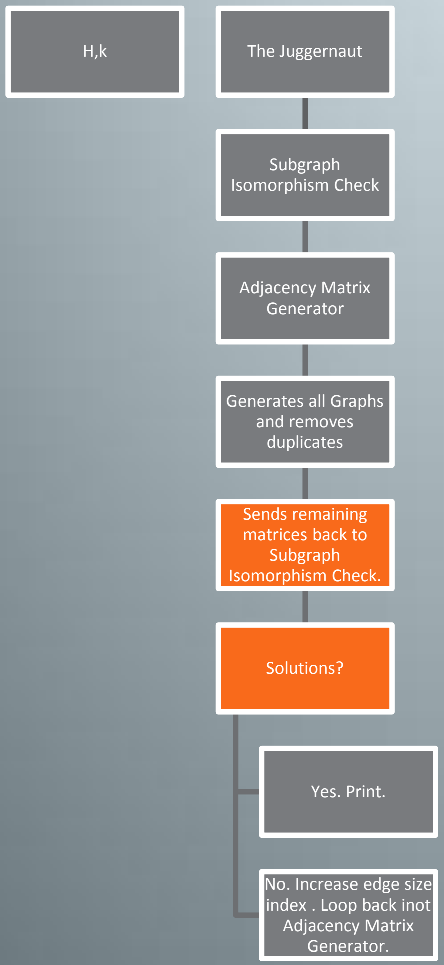
$$\begin{bmatrix} 0 & 1\cdot n+1 & 2\cdot n+1 & & (n-1)\cdot n+1 \\ & 0 & 2\cdot n+2 & \dots & (n-1)\cdot n+2 \\ & & 0 & & (n-1)\cdot n+3 \\ & & & \ddots & \vdots \\ & & & & (n-1)\cdot n+(n-1) \\ & & & & 0 \end{bmatrix}$$

How does it work?



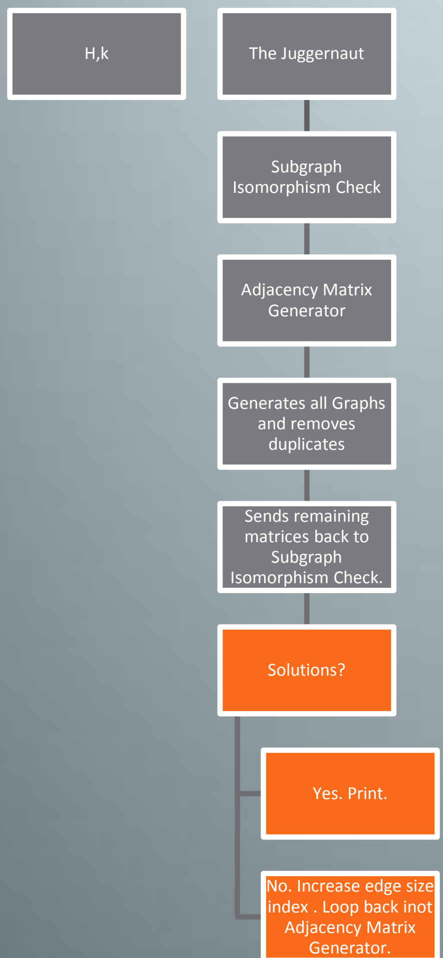
Adding this matrix to its transpose creates our full matrix. This process is repeated for each permutation. Matrices are trimmed of isolated vertices and similar matrices and isomorphisms are identified to reduce the number of possible solutions.

How does it work?



The remaining solutions are sent back to Subgraph Isomorphism Check, these remaining solutions are then checked against H by systematically removing vertices from our potential stable graphs. If a solution is found, it is stored.

How does it work?



If a solution is found, the remaining graphs on *e* edges are checked in case of multiple solutions. If no solutions are found, Subgraph Isomorphism Check returns a sad-face to The Juggernaut. The Juggernaut bumps up the edge index and the process is repeated until a solution is found.

Confirmed minimum $(P_n;k)$-stable Graphs?

Min $(P_n;k)$-stable	$k=1$	2	3	4	5	6	7	8	9	10
$n=2$	?	?	?	?	?	?	?	?	?	?
3	?	?	?	?	?	?	?	?	?	?
4	?	?	?	?	?	?	?	?	?	?
5	?	?	?	?	?	?	?	?	?	?
6	?	?	?	?	?	?	?	?	?	?
7	?	?	?	?	?	?	?	?	?	?
8	?	?	?	?	?	?	?	?	?	?
9	?	?	?	?	?	?	?	?	?	?
10	?	?	?	?	?	?	?	?	?	?

Confirmed minimum $(P_n; k)$-stable Graphs.

Min $(P_n;k)$-stable	$k=1$	2	3	4	5	6	7	8	9	10
$n=2$	$2\times P_2$	$3\times P_2$	$4\times P_2$	$5\times P_2$	$6\times P_2$	$7\times P_2$	$8\times P_2$	$9\times P_2$	$10\times P_2$	$11\times P_2$
3	$2\times P_3, C_4$									
4	C_5									
5										
6										
7										
8										
9										
10										

What's next?

- Improve The_JuggernautV2.
  - Introduce $k \geq 2$.
  - Find Isomorphism within the binary sequences.
  - Improve lower bound on edge set.
  - Remove the obvious bad graphs from the binary sequences.
- Fill out table.
  - Find some patterns for path stable graphs.
- Upgrade search to Cycles.
  - Petersen stable graphs?

To be continued…

Other Articles

- [1] On $(K_q; k)$-Stable Graphs
  - Andrzej Żak
- [2] On Vertex Stability with Regard to Complete Bipartite Subgraphs.
  - Aneta Dudek and Andrzej Żak
- [3] On $(K_q; k)$ Stable Graphs with small k
  - Jean-Luc Fouquet, Henri Thuillier, and Jean-Marie Vanherpe
- [4] On $(C_n; k)$ Stable Graphs
  - Sylwia Cichacz, Agnieszka Görlich, Magorzata Zwonek, and Andrzej Żak
- [5] Extremal P_4-stable Graphs
  - Illés Horváth and Gyula Y. Katona
- [6] On $(K_q; k)$ vertex stable graphs with minimum size
  - Jean-Luc Fouquet, Henri Thuillier, Jean-Marie Vanherpe, and A.P. Wojda
- [7] $(H; k)$ Stable Bipartite Graphs with Minimum Size
  - Aneta Dudek and Magorzata Zwonek
- [8] $(H; k)$ Stable Graphs with Minimum Size
  - Aneta Dudek, Artur Szymański and Magorzata Zwonek