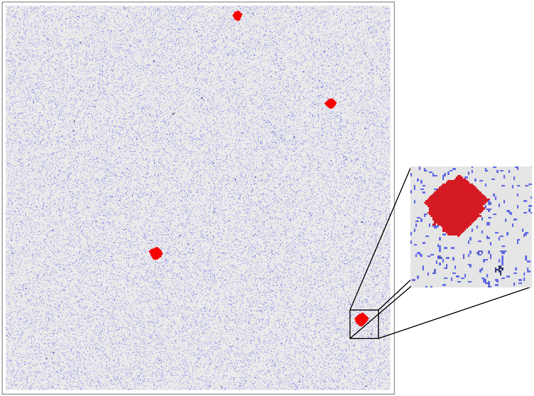
Deterministic percolation from random seeds

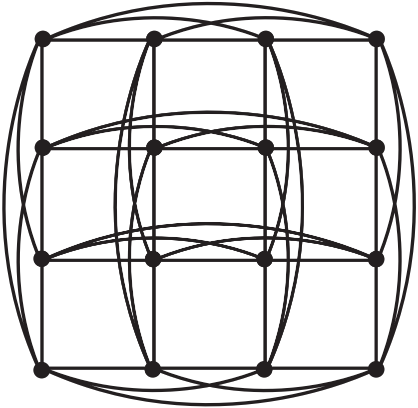


David Sivakoff

Ohio State University

November 17, 2014

Bootstrap Percolation

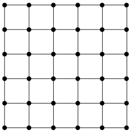


Graphs

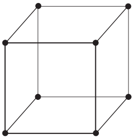
A graph, $G = (V, E)$, is a set of vertices, V, along with a set of undirected edges, $E \subset \binom{V}{2}$.

Examples:

- Square lattice (finite): $V = [n]^2 = \{1, 2, \ldots, n\}^2$,
  $E = \{(u, v) \in \binom{V}{2} \; : \; \|u - v\|_1 = 1\}$



- Hypercube: $V = \{0, 1\}^n$, $E = \{(u, v) \in \binom{V}{2} \; : \; \|u - v\|_1 = 1\}$.



Bootstrap Percolation

Fix a 'threshold' $\theta \in \mathbb{Z}_+$.

Let $\mathcal{N}(v)$ be the graph neighborhood of $v \in V$.

For $p = p(n) \in (0, 1)$, let $\{\omega(v)\}_{v \in V}$ be i.i.d. Bernoulli(p).

Bootstrap percolation is the increasing sequence of configurations in $\{0, 1\}^V$:

$$\omega_0 = \omega$$

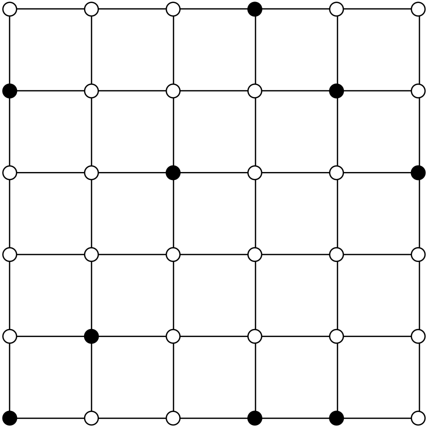
$$\omega_{j+1}(v) = \begin{cases} 1 & \text{if } \omega_j(v) = 1 \text{ or } \sum_{w \sim v} \omega_j(w) \geq \theta \\ 0 & \text{else} \end{cases}$$

for $k \geq 1$, and ω_∞ is the pointwise limit.

We say that ω_0 spans $F \subset V$ if $\omega_\infty|_F \equiv 1$, and spans G if $\omega_\infty \equiv 1$.

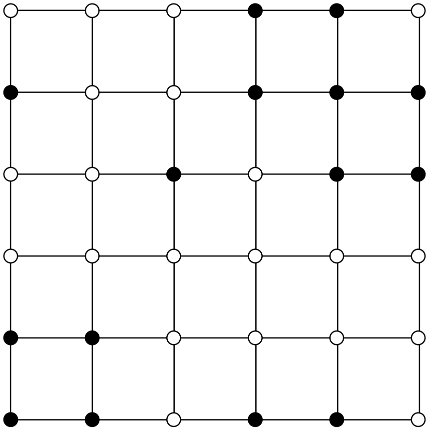
Bootstrap Percolation - Example

On the (finite) 2-dimensional nearest-neighbor lattice with $\theta = 2$.



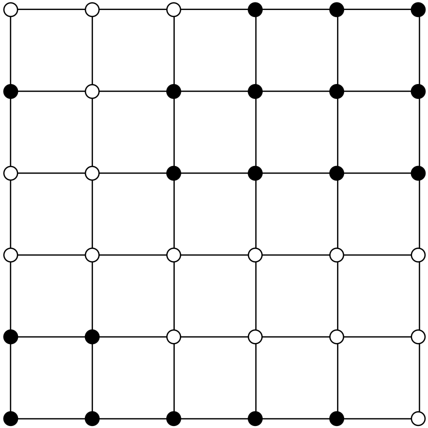
Bootstrap Percolation - Example

On the (finite) 2-dimensional nearest-neighbor lattice with $\theta = 2$.



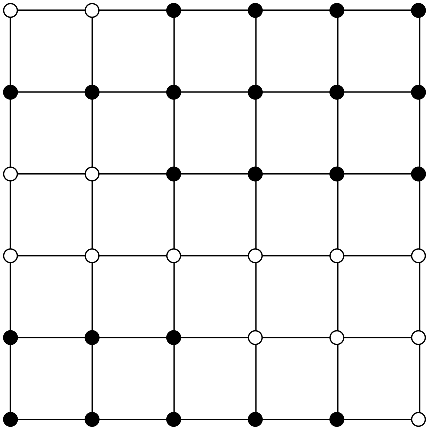
Bootstrap Percolation - Example

On the (finite) 2-dimensional nearest-neighbor lattice with $\theta = 2$.



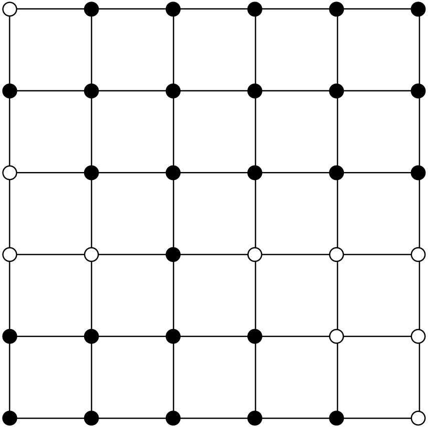
Bootstrap Percolation - Example

On the (finite) 2-dimensional nearest-neighbor lattice with $\theta = 2$.



Bootstrap Percolation - Example

On the (finite) 2-dimensional nearest-neighbor lattice with $\theta = 2$.



Bootstrap Percolation - History (Infinite Graphs)

Developed by Chalupa, Leith & Reich (1979) as a simple model of nucleation and metastability.

They proposed the model on the “Bethe lattice”, aka the infinite $(d+1)$-regular tree.

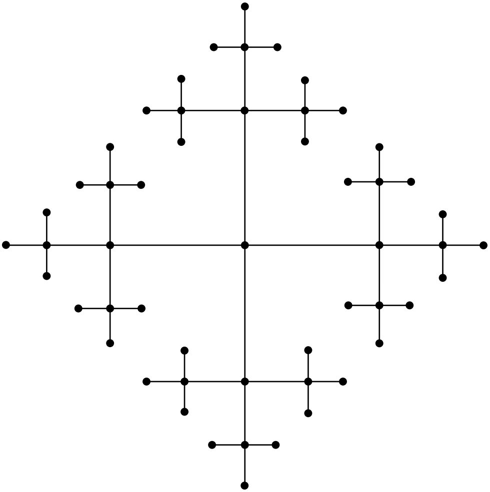
In the infinite graph setting,

$$p_c = p_c(G, \theta) := \inf\{p : \mathbb{P}_p(\omega_\infty \equiv 1) > 0\}.$$

For transitive graphs (e.g., trees, lattices),

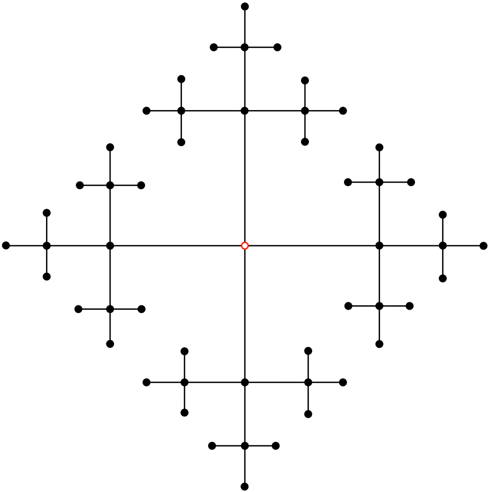
$$p_c = \sup\{p : \mathbb{P}_p(\omega_\infty \equiv 1) < 1\}.$$

Bootstrap Percolation - History (Infinite Graphs)



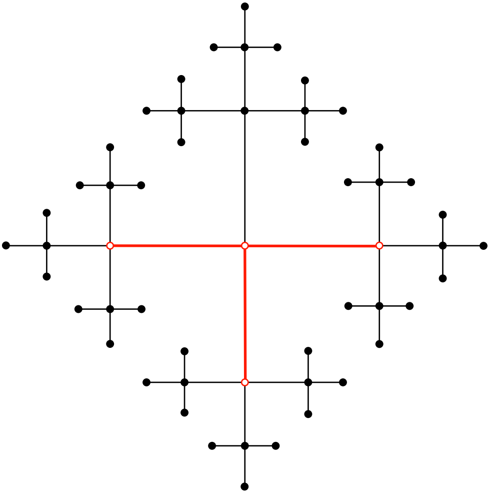
Consider
$(d+1) = (3+1)$-regular tree
and $\theta = 2$.

Bootstrap Percolation - History (Infinite Graphs)



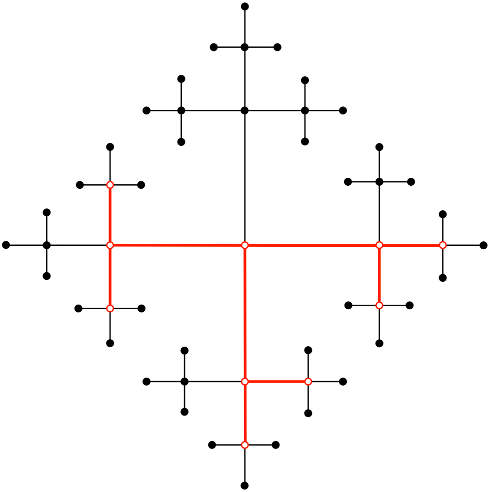
Consider
$(d+1) = (3+1)$-regular tree
and $\theta = 2$.

Bootstrap Percolation - History (Infinite Graphs)



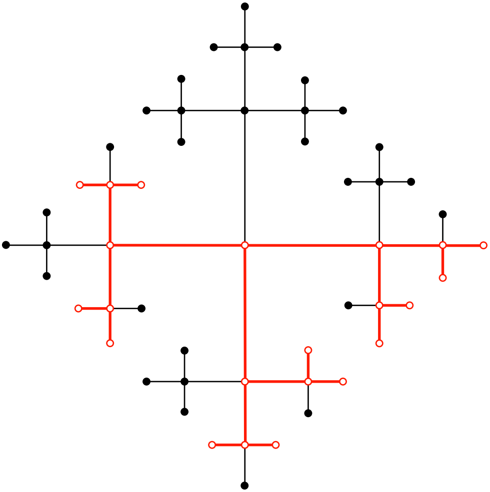
Consider
$(d+1) = (3+1)$-regular tree
and $\theta = 2$.

Bootstrap Percolation - History (Infinite Graphs)



Consider
$(d+1) = (3+1)$-regular tree
and $\theta = 2$.

Bootstrap Percolation - History (Infinite Graphs)



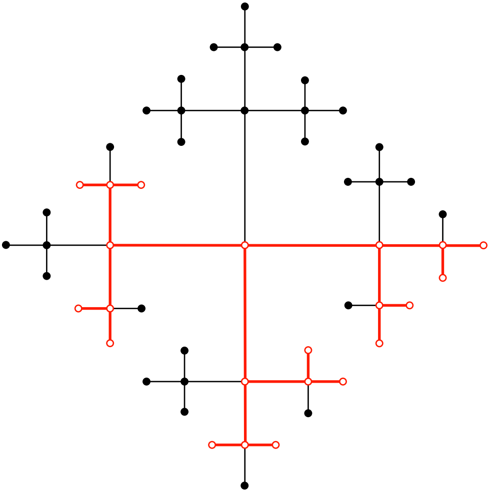
Consider
$(d + 1) = (3 + 1)$-regular tree
and $\theta = 2$.

$\mathbb{P}_p(\omega_\infty \neq 1) =$

$\mathbb{P}_p(\exists$ a 3-regular subtree

with no open vertices)

Bootstrap Percolation - History (Infinite Graphs)

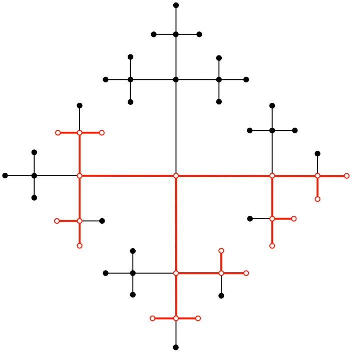


Consider
$(d+1) = (3+1)$-regular tree
and $\theta = 2$.

$\mathbb{P}_p(\omega_\infty \neq 1) =$
$\mathbb{P}_p(\exists$ a 3-regular subtree
with no open vertices)

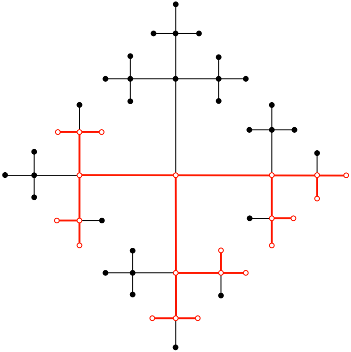
$= 1$
$\iff$ $\mathbb{P}_p$(empty binary tree rooted at v within ternary subtree rooted at $v) > 0$.

Bootstrap Percolation - History (Infinite Graphs)



$\mathbb{P}_p$(empty binary tree rooted at v within ternary subtree rooted at v) $=: y = y(p)$.

Bootstrap Percolation - History (Infinite Graphs)

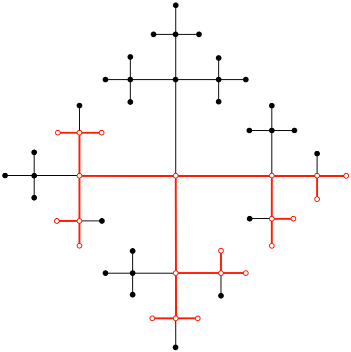


$\mathbb{P}_p$(empty binary tree rooted at v within ternary subtree rooted at v) $=: y = y(p)$.

$$y = \sum_{k=2}^{3} \binom{3}{k} [(1-p)y]^k [1-(1-p)y]^{3-k}$$

$$=: f(y, p).$$

Bootstrap Percolation - History (Infinite Graphs)

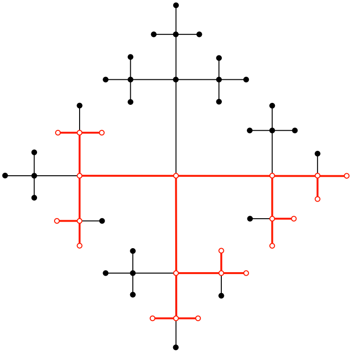


$\mathbb{P}_p$(empty binary tree rooted at v within ternary subtree rooted at v)
$=:y= y(p)$.

$$y = \sum_{k=2}^{3} \binom{3}{k} [(1-p)y]^k [1-(1-p)y]^{3-k}$$
$$=: f(y,p).$$

0 is always a solution, but can show that y is the largest solution in $[0,1]$.

Bootstrap Percolation - History (Infinite Graphs)



$\mathbb{P}_p$(empty binary tree rooted at v within ternary subtree rooted at v) $=: y = y(p)$.

$$y = \sum_{k=2}^{3} \binom{3}{k} [(1-p)y]^k [1-(1-p)y]^{3-k}$$
$$=: f(y,p).$$

0 is always a solution, but can show that y is the largest solution in $[0,1]$.

$$p_c = \inf\{p : y = f(y,p) \text{ has only one root at } y = 0\}$$
$$= \frac{1}{9}$$

Bootstrap Percolation - History (Infinite Graphs)

Most interest is in lattices, such as $\mathbb{Z}^d$.

First rigorous result is due to van Enter (1987) for $d = 2$, extended to $d \geq 2$ by Schonmann (1992)

Theorem (Nearest-neighbor lattice $\mathbb{Z}^d$)

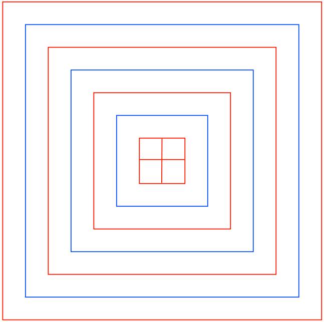
If $\theta \leq d$, then $p_c = 0$. If $\theta > d$, then $p_c = 1$.

Interesting cases are $2 \leq \theta \leq d$.

Bootstrap Percolation - History (Infinite Graphs)

When $d = \theta = 2$,

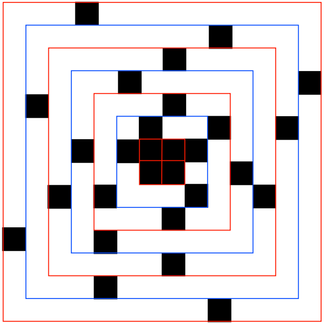
$$\mathbb{P}_p(\omega_\infty \equiv 1) \geq \mathbb{P}_p(\text{GOOD configuration exists}).$$



Bootstrap Percolation - History (Infinite Graphs)

When $d = \theta = 2$,

$$\mathbb{P}_p(\omega_\infty \equiv 1) \geq \mathbb{P}_p(\text{GOOD configuration exists}).$$



$$\mathbb{P}_p(\text{GOOD configuration at } 0) \geq \left[\prod_{k=1}^{\infty} 1 - (1-p)^k\right]^4$$

Bootstrap Percolation - History (Infinite Graphs)

$$
\begin{aligned}
\mathbb{P}_p(\text{GOOD configuration at } 0) &\geq \left[\prod_{k=1}^{\infty} 1-(1-p)^k\right]^4 \\
&= \left[\exp\left(\sum_{k=1}^{\infty} \log(1-(1-p)^k)\right)\right]^4 \\
&\geq \left[\exp\left(\frac{1}{p}\sum_{k=1}^{\infty} \log\left(1-e^{-kp}\right) p\right)\right]^4 \\
&\geq \exp\left(\frac{4}{p}\int_0^{\infty} \log\left(1-e^{-x}\right) dx\right) \\
&= \exp\left(-\frac{4}{p}\cdot\frac{\pi^2}{6}\right) \\
&> 0.
\end{aligned}
$$

□

Bootstrap Percolation - History (Finite Graphs)

For a finite box of side length n, how large does p need to be?

In finite graphs, the critical value $p_c = p_c(n)$ is defined as

$$\mathbb{P}_{p_c}(\omega_\infty \equiv 1) = 1/2.$$

Bootstrap Percolation - History (Finite Graphs)

Scaling for p_c:

Lattice cubes: $V = [n]^d \subset \mathbb{Z}^d$, nearest neighbor edges.

- $\theta = 2$: $(\log n)^{-(d-1)}$ [Aizenman & Lebowitz, 1988].

Hypercube: $V = \{0, 1\}^n$, nearest neighbor edges.

Bootstrap Percolation - History (Finite Graphs)

Scaling for p_c:

Lattice cubes: $V = [n]^d \subset \mathbb{Z}^d$, nearest neighbor edges.

- $\theta = 2$: $(\log n)^{-(d-1)}$ [Aizenman & Lebowitz, 1988].
- $3 \leq \theta \leq d$: $(\log^{\theta-1} n)^{-(d-\theta+1)}$ [Cerf & Cirillo, 1999; Cerf & Manzo, 2002].

Hypercube: $V = \{0, 1\}^n$, nearest neighbor edges.

Bootstrap Percolation - History (Finite Graphs)

Scaling for p_c:

Lattice cubes: $V = [n]^d \subset \mathbb{Z}^d$, nearest neighbor edges.

- $\theta = 2$: $(\log n)^{-(d-1)}$ [Aizenman & Lebowitz, 1988].
- $3 \leq \theta \leq d$: $(\log^{\theta-1} n)^{-(d-\theta+1)}$ [Cerf & Cirillo, 1999; Cerf & Manzo, 2002].

Hypercube: $V = \{0, 1\}^n$, nearest neighbor edges.

- $\theta = 2$: $n^{-2} 2^{-2\sqrt{n}}$ [Balogh & Bollobás, 2006]

Bootstrap Percolation - History (Finite Graphs)

Scaling for p_c:

Lattice cubes: $V = [n]^d \subset \mathbb{Z}^d$, nearest neighbor edges.

- $\theta = 2$: $(\log n)^{-(d-1)}$ [Aizenman & Lebowitz, 1988].
- $3 \leq \theta \leq d$: $(\log^{\theta-1} n)^{-(d-\theta+1)}$ [Cerf & Cirillo, 1999; Cerf & Manzo, 2002].

Hypercube: $V = \{0,1\}^n$, nearest neighbor edges.

- $\theta = 2$: $n^{-2}2^{-2\sqrt{n}}$ [Balogh & Bollobás, 2006]
- $\theta = \lceil n/2 \rceil$: $\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\log n}{n}}$ [Balogh, Bollobás, & Morris, 2009]

Bootstrap Percolation - History (Finite Graphs)

Scaling for p_c:

Lattice cubes: $V = [n]^d \subset \mathbb{Z}^d$, nearest neighbor edges.

- $\theta = 2$: $(\log n)^{-(d-1)}$ [Aizenman & Lebowitz, 1988].
- $3 \leq \theta \leq d$: $(\log^{\theta-1} n)^{-(d-\theta+1)}$ [Cerf & Cirillo, 1999; Cerf & Manzo, 2002].

Hypercube: $V = \{0,1\}^n$, nearest neighbor edges.

- $\theta = 2$: $n^{-2}2^{-2\sqrt{n}}$ [Balogh & Bollobás, 2006]
- $\theta = \lceil n/2 \rceil$: $\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\log n}{n}}$ [Balogh, Bollobás, & Morris, 2009]

These results suggest existence of an *order parameter:* a function of n and p whose size determines whether $\mathbb{P}_p(\omega_\infty \equiv 1)$ is near 0 or 1.

Bootstrap Percolation - History (Finite Graphs)

Does $\mathbb{P}_p(\omega_\infty \equiv 1)$ exhibit a sharp jump from 0 to 1 as the order parameter increases?

If so, does the location of the sharp jump converge?

Bootstrap Percolation - History (Finite Graphs)

- For $V = [n]^2$ and $\theta = 2$: $p_c \sim \frac{\pi^2}{18 \log n}$ [Holroyd, 2003]

Bootstrap Percolation - History (Finite Graphs)

- For $V = [n]^2$ and $\theta = 2$: $p_c \sim \frac{\pi^2}{18 \log n}$ [Holroyd, 2003]

- For $V = [n]^d$ and $2 \leq \theta \leq d$, sharp threshold established by [Balogh, Bollobás, Duminil-Copin, & Morris, 2012]

Bootstrap Percolation - History (Finite Graphs)

- For $V = [n]^2$ and $\theta = 2$: $p_c \sim \frac{\pi^2}{18 \log n}$ [Holroyd, 2003]

- For $V = [n]^d$ and $2 \le \theta \le d$, sharp threshold established by [Balogh, Bollobás, Duminil-Copin, & Morris, 2012]

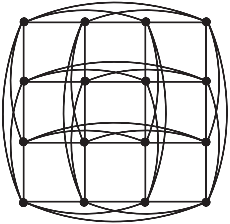
- For $V = [n]^2$, $\theta = 2$, and the 'cross' neighborhood ($k - 1$ nearest points in each of 4 directions): $p_c \sim \frac{\pi^2}{3k(k+1) \log n}$ [Holroyd, Liggett & Romik, 2004]

The Hamming Graph

The Hamming graph with side length n and dimension d is the graph with the following vertex set, V, and edge set, E.

$$V = \{1, 2, \ldots, n\}^d$$
$$E = \{(x, y) \in V \times V : d(x, y) = 1\},$$

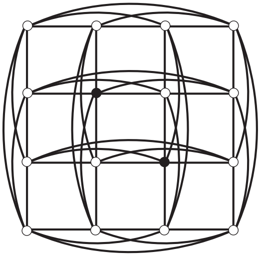
where $d(x, y)$ is the Hamming distance between x and y (number of coordinates at which they differ). Denote this graph by $H = H(d, n)$



Main Questions

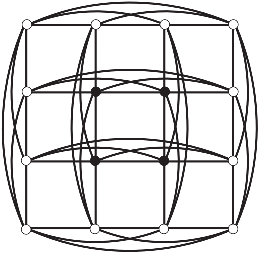
- Fix d and θ. For which values of $p = p(n)$ does $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1$? For which values of p does $\mathbb{P}_p(\omega_\infty \equiv 1) \to 0$?
- Does the probability of spanning H converge to a nontrivial limit in some regime? If so, what is the limit?

Test case: $\theta = 2$, $d = 2$



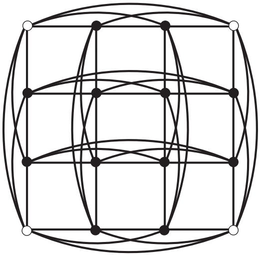
When $d = 2$, two (non-collinear) open vertices are necessary and sufficient for spanning.

Test case: $\theta = 2$, $d = 2$



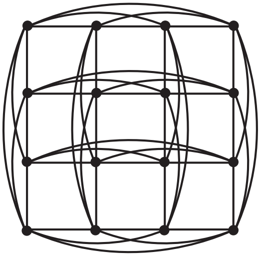
When $d = 2$, two (non-collinear) open vertices are necessary and sufficient for spanning.

Test case: $\theta = 2$, $d = 2$



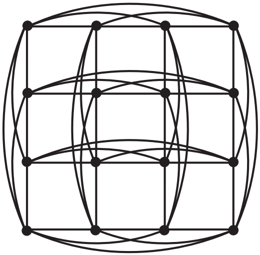
When $d = 2$, two (non-collinear) open vertices are necessary and sufficient for spanning.

Test case: $\theta = 2$, $d = 2$



When $d = 2$, two (non-collinear) open vertices are necessary and sufficient for spanning.

Test case: $\theta = 2$, $d = 2$



When $d = 2$, two (non-collinear) open vertices are necessary and sufficient for spanning.

If $p = an^{-2}$ then $|\omega_0| \implies \text{Poisson}(a)$

This implies $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - (1 + a)e^{-a}$.

Case: $d = 2$, $\theta \geq 3$

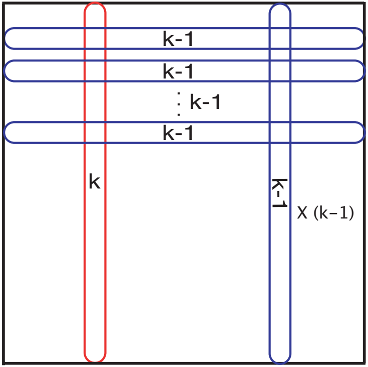
Theorem (Gravner, Hoffman, Pfeiffer, S. (2014))

Let $k \geq 2$ *and* $p = a \cdot n^{-(k+1)/k}$.

- *If* $\theta = 2k - 1$, *then* $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - \exp(-2a^k/k!)$.
- *If* $\theta = 2k$, *then* $\mathbb{P}_p(\omega_\infty \equiv 1) \to \left[1 - \exp(-a^k/k!)\right]^2$.

Case: $d = 2$, $\theta \geq 3$

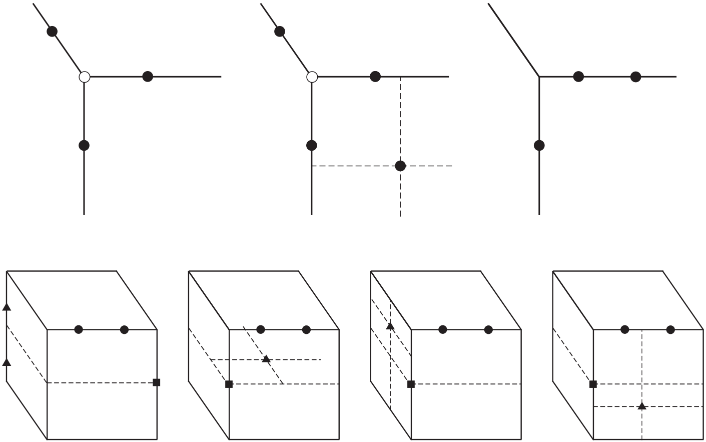
"Proof" for $\theta = 2k - 1$:



One line with k open vertices is likely to span. The number of lines with k open vertices converges to a Poisson($2a^k/k!$) r.v.

Case: $d = 3$, $\theta = 3$

When $p = an^{-2}$ these configurations contribute to $\{\omega_0 \text{ spans H}\}$.



Case: $d = 3$, $\theta = 3$

Theorem (Gravner, Hoffman, Pfeiffer, S. (2014))

If $p = an^{-2}$ then as $n \to \infty$

$$\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - e^{-a^3-(3/2)a^2(1-e^{-2a})} \times \left[\frac{3}{2}a^2\left(\left(e^{-a} + ae^{-3a}\right)^2 - e^{-2a}\right) e^{-a^2 e^{-2a}} + e^{a^3 e^{-3a}}\right].$$

Critical exponents for other d and θ

When $p \asymp n^{-\alpha}$ it is easy to show:

- If $\alpha > 1 + \frac{d}{\theta}$ then ω_0 will not span H w.h.p.
  (All vertices have $< \theta$ open neighbors.)

Critical exponents for other d and θ

When $p \asymp n^{-\alpha}$ it is easy to show:

- If $\alpha > 1 + \frac{d}{\theta}$ then ω_0 will not span H w.h.p.
  (All vertices have $< \theta$ open neighbors.)
- If $\alpha \leq 1$ then ω_0 will span H w.h.p.
  (Each vertex has $\geq \theta$ open neighbors with positive probability.)

Critical exponents for other d and θ

Recall $p_c = p_c(d, \theta, n)$ is such that:

$$\mathbb{P}_{p_c}(\omega_\infty \equiv 1) = 1/2.$$

Theorem: Critical exponents for large θ

For fixed $d \geq 3$, θ sufficiently large depending on d, and n sufficiently large depending on d, θ,

$$1 + \frac{2}{\theta} + \frac{\sqrt{7}}{\theta^{3/2}} \leq \frac{-\log p_c}{\log n} \leq 1 + \frac{2}{\theta} + \frac{4(d^2+1)}{\theta^{3/2}}.$$

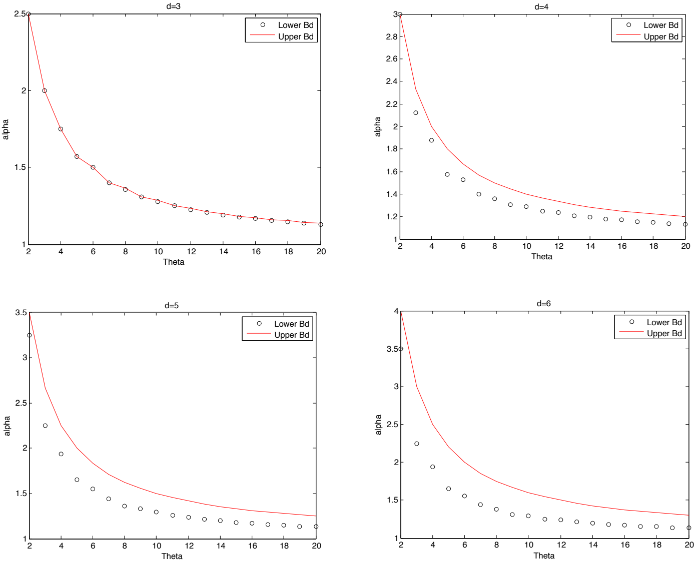
Bounds on critical exponents

For $d = 3$, we have matching bounds for some small values of θ

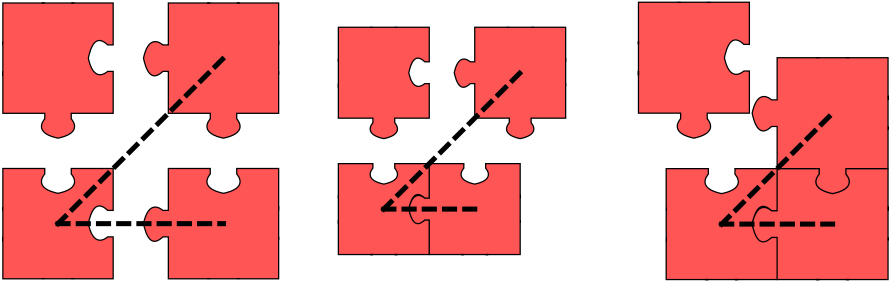
θ	2	3	4	5	6	7	8	9	10	11	12
Lower Bound	$\frac{5}{2}$	2	$\frac{7}{4}$	$\frac{11}{7}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{19}{14}$	$\frac{17}{13}$	$\frac{23}{18}$	$\frac{5}{4}$	$\frac{27}{22}$
Upper Bound	$\frac{5}{2}$	2	$\frac{7}{4}$	$\frac{11}{7}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{15}{11}$	$\frac{17}{13}$	$\frac{9}{7}$	$\frac{5}{4}$	$\frac{21}{17}$

- Lower bound is via dimension reduction.
- Upper bound is the minimum of $1 + \frac{3}{\theta}$ and either $1 + \frac{8}{3\theta - 1}$ if θ is odd or $\frac{8}{3\theta - 2}$ if θ is even.

Bounds on critical exponents



Jigsaw Percolation



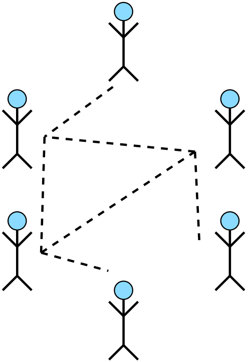
Joint work with: Charles Brummitt, Shirshendu Chatterjee, Partha Dey, Janko Gravner

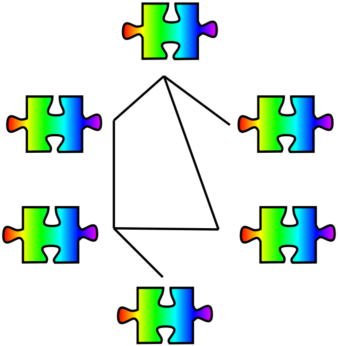
Motivation

- DARPA challenge: UCSD team used crowdsourcing to piece together shredded paper.
- Polymath Project: Tim Gowers' experiment with "massively collaborative mathematics."
- How might people *cooperatively* combine their individual ideas to solve a problem?

People and Ideas (Puzzle Pieces)

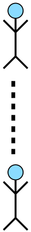
A new dynamic on multitype networks





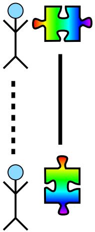
People and Ideas (Puzzle Pieces)

- If two people know each other...



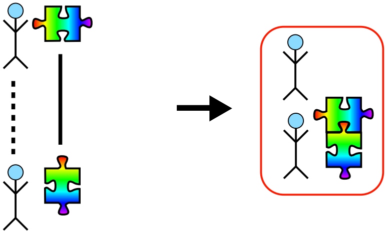
People and Ideas (Puzzle Pieces)

- If two people know each other...
- and have compatible ideas...



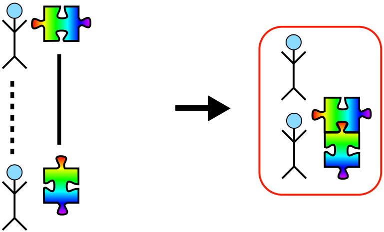
People and Ideas (Puzzle Pieces)

- If two people know each other...
- and have compatible ideas...
- then they merge their ideas.

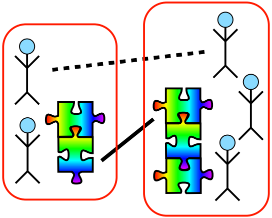


People and Ideas (Puzzle Pieces)

- If two people know each other...
- and have compatible ideas...
- then they merge their ideas.

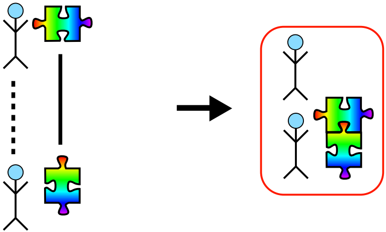


- Generally, if two groups with merged ideas know each other and have compatible ideas...

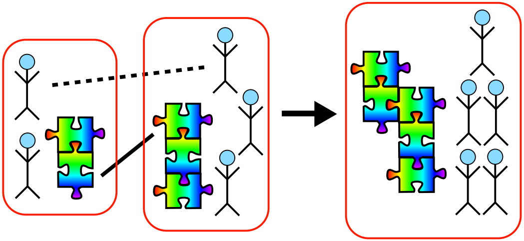


People and Ideas (Puzzle Pieces)

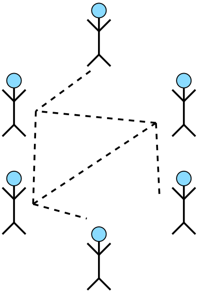
- If two people know each other...
- and have compatible ideas...
- then they merge their ideas.

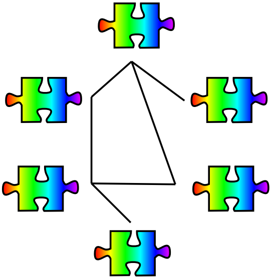


- Generally, if two groups with merged ideas know each other and have compatible ideas...
- then they merge their ideas.



Jigsaw Percolation Model

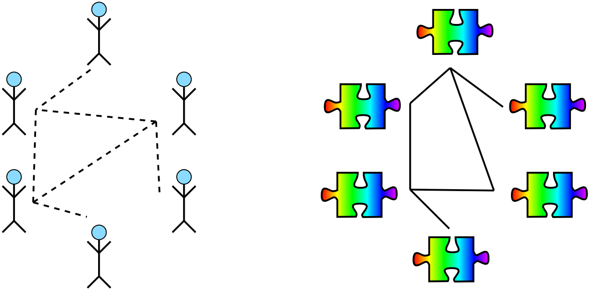




People Graph of who knows whom: (V, E_{people}).

Puzzle Graph of compatible ideas:

Jigsaw Percolation Model

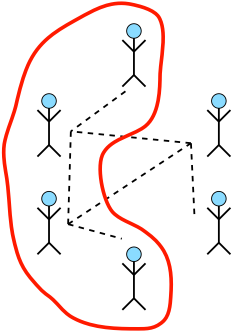


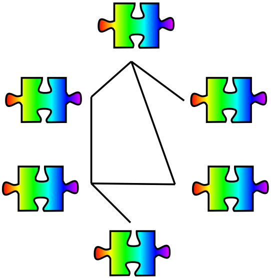
Each person has one unique piece of the puzzle.

People Graph of who knows whom: (V, E_{people}).

Puzzle Graph of compatible ideas: (V, E_{puzzle}).

Jigsaw Percolation Model





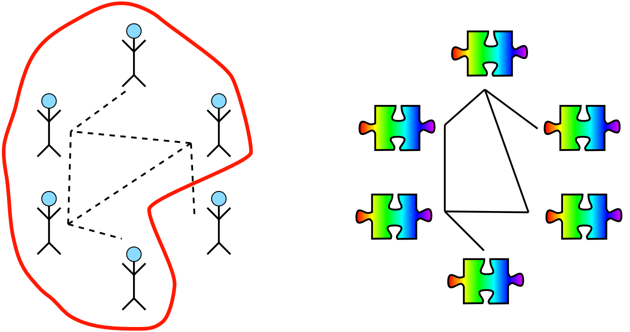
Each person has one unique piece of the puzzle.

People Graph of who knows whom: (V, E_{people}).

Puzzle Graph of compatible ideas: (V, E_{puzzle}).

Successively merge groups that know one another and have compatible puzzle pieces.

Jigsaw Percolation Model



Each person has one unique piece of the puzzle.

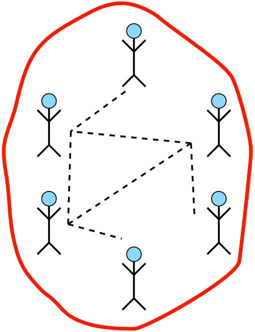
People Graph of who knows whom: (V, E_{people}).

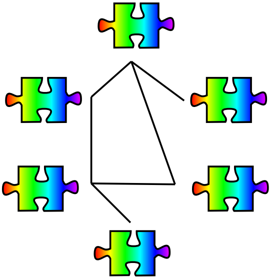
Puzzle Graph of compatible ideas: (V, E_{puzzle}).

Successively merge groups that know one another and have compatible puzzle pieces.

Jigsaw Percolation Model

Solved the puzzle!





Each person has one unique piece of the puzzle.

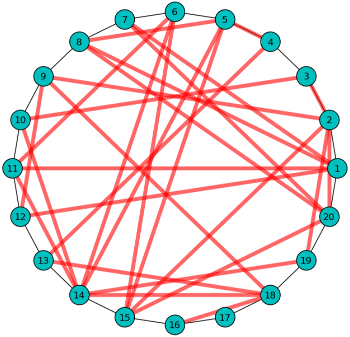
People Graph of who knows whom: (V, E_{people}).

Puzzle Graph of compatible ideas: (V, E_{puzzle}).

Successively merge groups that know one another and have compatible puzzle pieces.

Jigsaw Percolation Model

- People Graph: Erdős-Rényi random graph $(V, E_{\text{people}}) \sim G(n, p)$.
- Puzzle Graphs: Connected graphs on n vertices.



How connected must the people graph be to solve the puzzle?

Results

Theorem (Brummitt, Chatterjee, Dey, S. (2014))

For any connected puzzle graph if $p = \lambda / \log n$ with $\lambda > \pi^2/6$, then as $n \to \infty$

$$\mathbb{P}_p(\texttt{Solve}) \to 1.$$

Theorem (Gravner, S.)

Let D be the maximum puzzle degree. If $p = \mu/(D \log n)$ with $\mu < 2e^{-4}$, then

$$\mathbb{P}_p(\texttt{Solve}) \to 0.$$

Results

Theorem (Brummitt, Chatterjee, Dey, S. (2014))

For any connected puzzle graph if $p = \lambda / \log n$ with $\lambda > \pi^2/6$, then as $n \to \infty$

$$\mathbb{P}_p(\texttt{Solve}) \to 1.$$

Theorem (Gravner, S.)

Let D be the maximum puzzle degree. If $p = \mu/(D \log n)$ with $\mu < 2e^{-4}$, then

$$\mathbb{P}_p(\texttt{Solve}) \to 0.$$

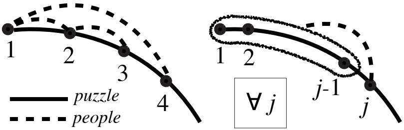
Corollary: For puzzles of bounded degree, $p_c = \Theta(1/\log n)$.

Proofs (Main Ideas)

Upper Bound: $p_c \leq \frac{\pi^2}{6 \log n}$

Sufficient condition:

j is people-adjacent to $\{1, 2, \ldots, j-1\}$ for all j.

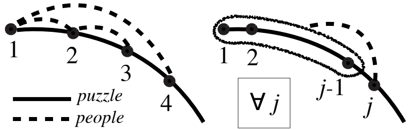


Proofs (Main Ideas)

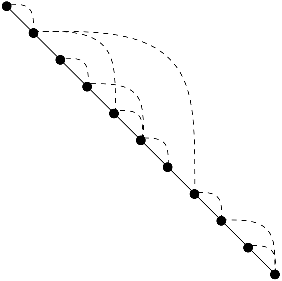
Upper Bound: $p_c \leq \frac{\pi^2}{6 \log n}$

Sufficient condition:

j is people-adjacent to $\{1, 2, \ldots, j-1\}$ for all j.



Lower Bound: $p_c \geq \frac{C}{D \log n}$



Necessary condition:

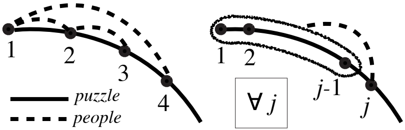
For any k there is a puzzle-connected set of size $\in [k, 2k]$ that is internally solved.

Proofs (Main Ideas)

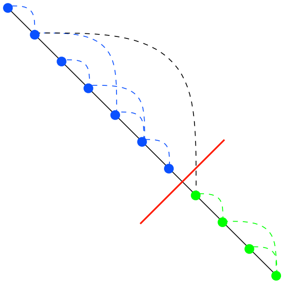
Upper Bound: $p_c \leq \frac{\pi^2}{6 \log n}$

Sufficient condition:

j is people-adjacent to $\{1, 2, \ldots, j-1\}$ for all j.



Lower Bound: $p_c \geq \frac{C}{D \log n}$



Necessary condition:

For any k there is a puzzle-connected set of size $\in [k, 2k]$ that is internally solved.

General Results: Improved Upper Bound

Matching order bounds for some famous D-regular transitive graphs: $p_c \asymp 1/(D \log N)$.

Theorem (Gravner, S.)

Let $p_c = \inf\{p : \mathbb{P}_p(\texttt{Solve}) > 1/2\}$.

- The d-dimensional torus $\mathbb{Z}_n^d$ has $p_c = \Theta(1/(d^2 \log n))$.

General Results: Improved Upper Bound

Matching order bounds for some famous D-regular transitive graphs: $p_c \asymp 1/(D \log N)$.

Theorem (Gravner, S.)

Let $p_c = \inf\{p : \mathbb{P}_p(\texttt{Solve}) > 1/2\}$.

- The d-dimensional torus $\mathbb{Z}_n^d$ has $p_c = \Theta(1/(d^2 \log n))$.
- The 2-dimensional torus $\mathbb{Z}_n^2$ with spread-out edges, $x \leftrightarrow y$ if $\|x - y\|_\infty \leq r$, has $p_c = \Theta(1/(r^2 \log n))$.

General Results: Improved Upper Bound

Matching order bounds for some famous D-regular transitive graphs: $p_c \asymp 1/(D \log N)$.

Theorem (Gravner, S.)

Let $p_c = \inf\{p : \mathbb{P}_p(\texttt{Solve}) > 1/2\}$.

- The d-dimensional torus $\mathbb{Z}_n^d$ has $p_c = \Theta(1/(d^2 \log n))$.
- The 2-dimensional torus $\mathbb{Z}_n^2$ with spread-out edges, $x \leftrightarrow y$ if $\|x - y\|_\infty \leq r$, has $p_c = \Theta(1/(r^2 \log n))$.
- The hypercube $\{0, 1\}^n$ has $p_c = \Theta(1/n^2)$.

General Results: Improved Upper Bound

Matching order bounds for some famous D-regular transitive graphs: $p_c \asymp 1/(D \log N)$.

Theorem (Gravner, S.)

Let $p_c = \inf\{p : \mathbb{P}_p(\texttt{Solve}) > 1/2\}$.

- The d-dimensional torus $\mathbb{Z}_n^d$ has $p_c = \Theta(1/(d^2 \log n))$.
- The 2-dimensional torus $\mathbb{Z}_n^2$ with spread-out edges, $x \leftrightarrow y$ if $\|x - y\|_\infty \leq r$, has $p_c = \Theta(1/(r^2 \log n))$.
- The hypercube $\{0, 1\}^n$ has $p_c = \Theta(1/n^2)$.
- The Hamming graph $\mathbb{Z}_n^d$ with edges $x \leftrightarrow y$ if $\|x - y\|_{\mathrm{H}} = 1$ has $p_c = \Theta(1/(d^2 n \log n))$.

Counterexamples

From this one might conjecture that $p_c \asymp 1/(D \log N)$ for transitive D-regular graphs.

Counterexamples

From this one might conjecture that $p_c \asymp 1/(D \log N)$ for transitive D-regular graphs.

Simple counterexample: $G_{\text{puz}} = K_n$, so $D \sim n$.
If $1/(n \log n) \ll p \ll \log n/n$, then G_{ppl} is disconnected whp, so $\mathbb{P}_p(\texttt{Solve}) \to 0$.

Counterexamples

From this one might conjecture that $p_c \asymp 1/(D \log N)$ for transitive D-regular graphs.

Simple counterexample: $G_{\text{puz}} = K_n$, so $D \sim n$.
If $1/(n \log n) \ll p \ll \log n/n$, then G_{ppl} is disconnected whp, so $\mathbb{P}_p(\texttt{Solve}) \to 0$.

Nontrivial counterexample: $G_{\text{puz}} = K_n \times \mathbb{Z}_{(\log n)^3}$, where $\mathbb{Z}_m$ is the cycle of m vertices, so $D \sim n$, $N = n(\log n)^3$.
If $p = \mu/(n \log n)$ with $\mu > 0$, then G_{ppl} is connected whp, but $\mathbb{P}_p(\texttt{Solve}) \to 0$.

Model Generalization

Fix thresholds: $\theta, \tau, \sigma \geq 1$.

$\texttt{links}(v, A) = \#$ Puzzle edges between v and $A \subset V$.
$\texttt{collaborators}(v, A) = \#$ People edges between v and $A \subset V$.

New Rules

Merge two clusters, W_1 and W_2, if at least one of the following hold:

(1) there are doubly connected vertices $v_1 \in W_1$ and $v_2 \in W_2$;
(2) there is a vertex $v_1 \in W_1$ with $\texttt{collaborators}(v_1, W_2) \geq \sigma$ and $\texttt{links}(v_1, W_2) \geq \tau$.
(3) there is a vertex $v_1 \in W_1$ with $\texttt{links}(v_1, W_2) \geq \theta$;

Model Generalization

Fix thresholds: $\theta, \tau, \sigma \geq 1$.

$\texttt{links}(v, A) = \#$ Puzzle edges between v and $A \subset V$.
$\texttt{collaborators}(v, A) = \#$ People edges between v and $A \subset V$.

New Rules

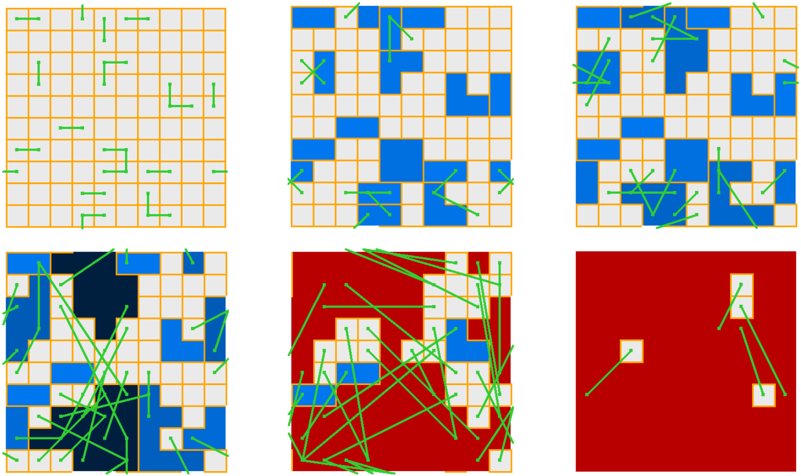
Merge two clusters, W_1 and W_2, if at least one of the following hold:

(1) there are doubly connected vertices $v_1 \in W_1$ and $v_2 \in W_2$;
(2) there is a vertex $v_1 \in W_1$ with $\texttt{collaborators}(v_1, W_2) \geq \sigma$ and $\texttt{links}(v_1, W_2) \geq \tau$.
(3) there is a vertex $v_1 \in W_1$ with $\texttt{links}(v_1, W_2) \geq \theta$;

If $\tau = \sigma = 1$ and $\theta = \infty$, this gives the Adjacent-Edge jigsaw percolation.

Open question: Is Adjacent-Edge jigsaw percolation distinguishable from basic jigsaw percolation?

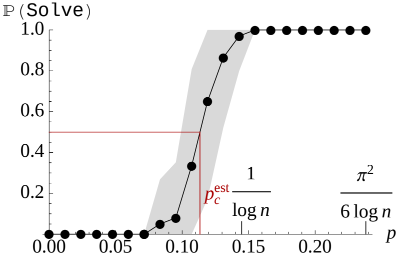
Model Generalization



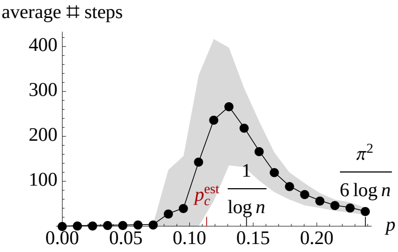
Adjacent-Edge JP on 10×10 torus, with $p = 0.11$, at times $t = 0, \ldots, 5$.

Sharp Transitions for the Ring Puzzle

Adjacent-Edge JP on $\mathbb{Z}_n$ with $n = 1000$, averaged over 200 trials.



Probability of `Solve`.



Time to final configuration.

Sharp Transitions for the Ring Puzzle

Theorem: Threshold for solving the ring.

Let $\sigma \geq 1$, $\tau = 1$, $\theta = \infty$ and

$$\lambda_c := -\int_0^\infty \log \mathbb{P}(\mathrm{Poisson}(x) \geq \sigma)\, dx.$$

If $G_{\text{puz}} = \mathbb{Z}_n$ then

$$p_c \log n \to \lambda_c,$$

with sharp transition.

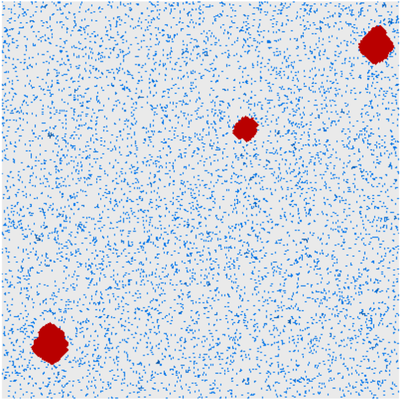
Theorem: Time to final configuration.

If $p \sim \lambda / \log n$ and T_f is the first time the final partition is reached, then, in probability

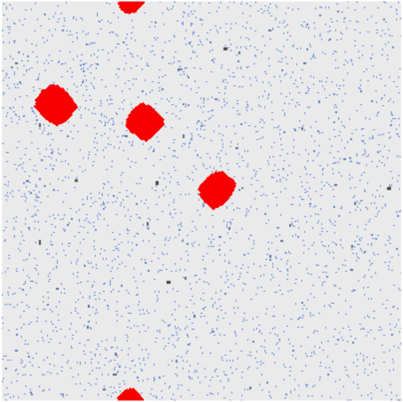
$$\begin{cases} \limsup_{n\to\infty} \frac{T_f}{\log n} < \infty & \text{if } \lambda < \lambda_c \\ \lim_{n\to\infty} \frac{\log T_f}{\log n} = \frac{\lambda_c}{\lambda} & \text{if } \lambda > \lambda_c \end{cases}$$

Jigsaw Percolation on $\mathbb{Z}_n^2$

$n = 400.$



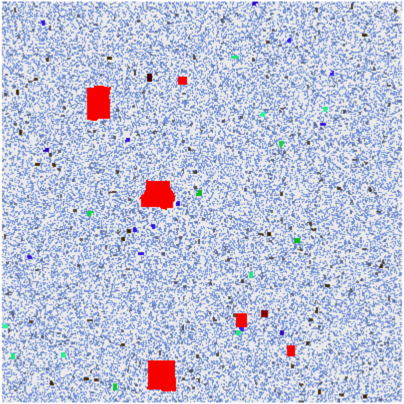
Adjacent-Edge BP at $t = 31$ with $p = 0.021$.



$\sigma = \tau = 1$ and $\theta = 2$. At $t = 31$ with $p = 0.009$.

Jigsaw Percolation on $\mathbb{Z}_n^2$

$n = 400$.



$\sigma = 1, \tau = 2$ and $\theta = \infty$. At $t = 91$ with $p = 0.11$.

Jigsaw Percolation on $\mathbb{Z}_n^2$

Let $\tau = \sigma = 1$ and $\theta = \infty$ (Adjacent-Edge JP).

Theorem: 2d-torus bounds

For all large enough n,

$$\frac{0.0388}{\log n} < p_c < \frac{0.303}{\log n}.$$

Proof Ideas:

- Lower Bound: Number of connected subsets of size k containing the origin is $\leq (4.65)^k$ [Finch '99].
- Upper Bound: Internally solve triangles and $p_c^{\text{site}} < 0.6795$ [Wierman '95].

Jigsaw Percolation on $\mathbb{Z}_n^2$

Let $\tau = 1, \sigma \geq 1$ and $\theta = 2$.

Theorem: 2d-torus jigsaw-bootstrap percolation

Let $g(x) = -\log(1 - e^{-x})$ and define

$$\lambda_c = \int_0^\infty g\left(\frac{x^{2\sigma+1}}{\sigma!}\right)\, dx = \frac{(\sigma!)^{1/(2\sigma+1)}\Gamma(\frac{1}{2\sigma+1})\zeta(\frac{2\sigma+2}{2\sigma+1})}{(2\sigma+1)}.$$

Then as $n \to \infty$,

$$p_c(\log n)^{2+\frac{1}{\sigma}} \to \lambda_c^{2+\frac{1}{\sigma}},$$

with sharp transition.

Note that σ affects the order of p_c!

Jigsaw Percolation on $\mathbb{Z}_n^2$

Let $\tau = 2, \sigma \geq 1$ and $\theta \geq 2$.

Theorem: 2d-torus restricted jigsaw percolation

If $\theta > 2$, then

$$\frac{\pi^2}{6} \leq \liminf_{n\to\infty} p_c \log n \leq \limsup_{n\to\infty} p_c \log n \leq \frac{\pi^2}{6} - \frac{1}{2}\int_0^\infty \log \mathbb{P}(\mathrm{Poisson}(x) \geq \sigma)\, dx.$$

If $\theta = 2$, then

$$p_c \log n \to \frac{\pi^2}{6}$$

with sharp transition.

Note that p_c does not depend on σ when $\theta = 2$.

Open problems

- For Adjacent-Edge BP on $\mathbb{Z}_n^2$, can a sharp transition be proved? For $\tau = 2, \theta > 2$?

Open problems

- For Adjacent-Edge BP on $\mathbb{Z}_n^2$, can a sharp transition be proved? For $\tau = 2, \theta > 2$?
- Thresholds for other puzzle graphs? ($\mathbb{Z}_n^d$, random regular graph, hypercube, Hamming graph, etc.)

Open problems

- For Adjacent-Edge BP on $\mathbb{Z}_n^2$, can a sharp transition be proved? For $\tau = 2, \theta > 2$?
- Thresholds for other puzzle graphs? ($\mathbb{Z}_n^d$, random regular graph, hypercube, Hamming graph, etc.)
- Rates of convergence?

Thank you!